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Mathematical Reviews

Edited by

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J. V. Wehausen, *Executive Editor*

Vol. 16, No. 2 February, 1955 pp. 103–206

The index issue for Vol. 15 will appear as Part 2 of one of the issues of Vol. 16.

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MATHEMATICAL REVIEWS

Published monthly, except August, by

THE AMERICAN MATHEMATICAL SOCIETY, Prince and Lemon Streets, Lancaster, Pennsylvania

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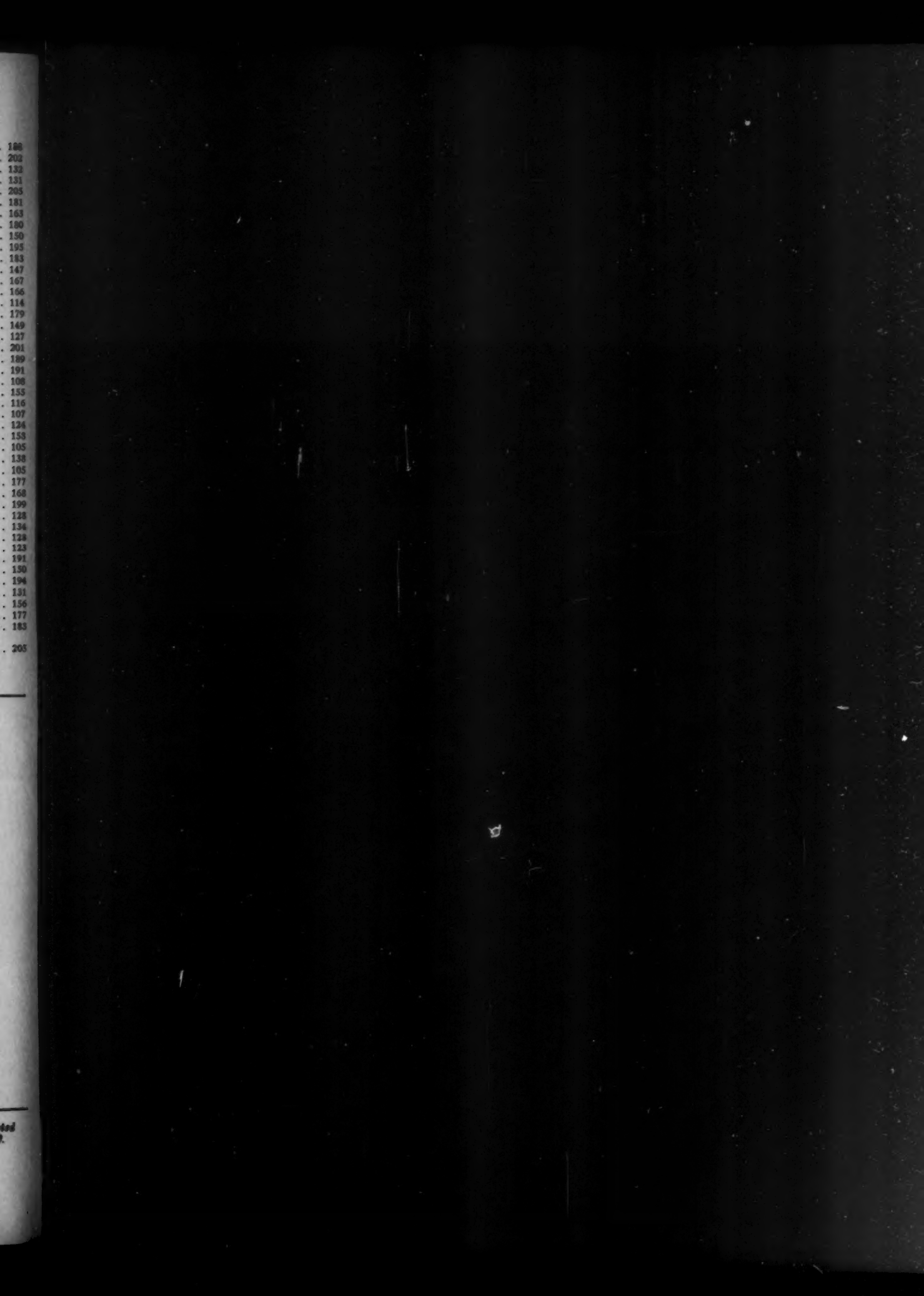
Editorial Office

MATHEMATICAL REVIEWS, 80 Waterman St., Providence 6, R. I.

Subscriptions: Price $20 per year ($10 per year to members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions may be addressed to MATHEMATICAL REVIEWS, Lancaster, Pennsylvania, but should preferably be addressed to the American Mathematical Society, 80 Waterman St., Providence 6, R. I.

This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. Its preparation is also supported currently under a contract with the Office of Scientific Research, Headquarters, Air Research and Development Command, U. S. Air Force. These organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Entered as second-class matter February 3, 1940 at the post office at Lancaster, Pennsylvania, under the act of March 3, 1879. Accepted for mailing at special rate of postage provided for in paragraph (d-2), section 34.40, P. L. and R. of 1948, authorized November 9, 1940.



Mathematical Reviews

Vol. 16, No. 2 FEBRUARY, 1955 Pages 103–206

FOUNDATIONS

Parry, William Tuthill. A new symbolism for the propositional calculus. J. Symbolic Logic 19, 161–168 (1954).

The symbolism for binary truth functions (tf) whose variables are p and q, is based on the following principle. A symbol consisting of four easily distinguished shapes is chosen, and each of these shapes is used to represent one of the basic tf which occur in Boolean expansions, namely $p \cdot q$, $p \cdot \bar{q}$, $\bar{p} \cdot q$, $\bar{p} \cdot \bar{q}$. Any tf (among the 12 remaining tf) is denoted by the symbol consisting of those shapes which represent the tf's which occur in its Boolean expansion.

G. Kreisel (Reading).

Standley, Gerald B. Ideographic computation in the propositional calculus. J. Symbolic Logic 19, 169–171 (1954).

Extension of Parry's symbolism [see the preceding review] to n-ary tf, and some examples. *G. Kreisel.*

Anderson, Alan Ross. Improved decision procedures for Lewis's calculus *S4* and von Wright's calculus *M*. J. Symbolic Logic 19, 201–214 (1954).

G. H. von Wright, in "An essay on modal logic" [North-Holland Publ. Co., Amsterdam, 1951; these Rev. 13, 614], sketched a decision method for his system M of modal logic. In this paper the author considerably simplifies this method. He gives a rigorous proof for the effectiveness of the analogous method for $S4$ and he states that the proof in the case of M can be given along the same lines. His method yields the decision for relatively simple formulas with remarkably little work. *A. Heyting* (Amsterdam).

Henkin, Leon. A generalization of the concept of ω-consistency. J. Symbolic Logic 19, 183–196 (1954).

Let (F) be an applied functional calculus, Γ a non-empty set of its individual constants. (F) is called (i) Γ^k-consistent if it contains no theorem $(\exists x_1)\cdots(\exists x_k)A(x_1, \cdots, x_k)$ without free variables such that, for each k-tuple $\gamma_1, \cdots, \gamma_k$ of Γ, $\neg A(\gamma_1, \cdots, \gamma_k)$ can be proved in (F), (ii) Γ^ω-consistent if it is Γ^k-consistent for all k. The calculus (F) is called Γ-satisfiable if it possesses a model whose individual elements belong to Γ. If Γ is finite, a Γ^1-consistent system is necessarily Γ-satisfiable. The author presents several systems whose only non-logical symbols are individual constants (the only predicate symbol is $=$) and, by selecting suitable subsets of these constants, shows: (I) for every $k \geqq 1$ there is a Γ^k-consistent system which is not Γ^{k+1}-consistent; (II) there is a Γ^ω-consistent system which is not Γ-satisfiable. He also observes that if (F) contains Hilbert's ϵ-symbol (or something equivalent to it) and if all ϵ-terms of (F) are included in Γ, then Γ^1-consistency ensures that (F) is Γ-satisfiable.

The relation to ω-consistency where 'Γ' consists of the numerals, is this: (I) has no analogue for any system of arithmetic in which, for every k, an enumeration $[\eta_1^k(n), \cdots, \eta_k^k(n)]$ of k-tuples of numerals is available, since then

$$(\exists x_1)\cdots(\exists x_k)A(x_1, \cdots, x_k) \leftrightarrow (\exists x)A[\eta_1^k(x), \cdots, \eta_k^k(x)].$$

An example of (II) is obtained as follows: a formula Q can be written down which states that an ω-inconsistency can be derived from 'Q' in Peano's arithmetic; then Peano's arithmetic with Q as an additional axiom is ω^ω-consistent, but Q is false (not ω-satisfiable). *G. Kreisel* (Reading).

Henkin, L. Boolean representation through propositional calculus. Fund. Math. 41, 89–96 (1954).

The paper establishes the equivalence between the Gödel-Malcev theorem (GM) [Malcev, Mat. Sbornik N.S. 1(43), 323–336 (1936)] and the boolean representation theorem (BR) [M. H. Stone, Trans. Amer. Math. Soc. 40, 37–111 (1936)] without using the axiom of choice. The known proofs of these two theorems use the axiom of choice. GM states that for every class of formulae which is consistent over the propositional calculus there exists a function φ taking the values T and F such that (i) for p of the set, $\varphi(p) = T$, (ii) for any propositional connective $\circ$, $\varphi(p \circ q) = \varphi(p) \circ \varphi(q)$. Given a boolean algebra, the author considers propositional symbols p_x, where x ranges over the elements of the algebra, and expresses the complete diagram of the algebra by means of formulae made up of p_x. These formulae are consistent, and suitable sets of their φ-functions yield a boolean algebra of sets which is isomorphic to the given algebra; this proves BR. The reverse implication was proved, in effect, by Rasiowa and Sikorski [Fund. Math. 38, 230–232 (1951); these Rev. 15, 385]. The author makes some observations on the relation between the cardinal of the algebra and the set of φ-functions.

G. Kreisel (Reading).

Łoś, J. Sur le théorème de Gödel pour les théories indénombrables. Bull. Acad. Polon. Sci. Cl. III. 2, 319–320 (1954).

This note establishes the equivalence between the Gödel-Malcev completeness theorem and the existence of prime ideals in Boolean algebra, without using the axiom of choice. The method is essentially the same as Henkin's [see the preceding review]. The author announces a proof that the Gödel-Malcev theorem is (effectively) equivalent to the ordering principle, but he does not know whether it is equivalent to the axiom of choice. *G. Kreisel.*

Jaśkowski, S. Example of a class of systems of ordinary differential equations having no decision method for existence problems. Bull. Acad. Polon. Sci. Cl. III. 2, 155–157 (1954).

Let $\epsilon_{ij} = \pm 1$, $1 \leqq i \leqq h$, $1 \leqq j \leqq k$, and let $f_j(a_1, \cdots, a_n)$ be functions obtained by superposition of

$$\psi(a, b)\ [= \tfrac{1}{2}(a+b)(a+b+1)+b+1].$$

Here $A\ (= A^1)$ denotes a set of non-negative integers, A^{-1} its complement. Pepis' reduction of the decision problem for the predicate calculus [Fund. Math. 30, 257–348 (1938)] shows that there is no procedure for deciding whether any given f_j and $\{\epsilon_{ij}\}$ possess the following property (P): for

every A there is an $i \leqq k$ such that for all a and all $j \leqq k$, $f_j(a_1, \cdots, a_n) \in A^{\epsilon_{ij}}$. The author exhibits a system of differential equations containing Pepis' functions f_j as parameters, and a boundary condition $B(\theta)$, expressed by a polynomial in the dependent variables with the coefficients ϵ_{ij}. He announces that this system possesses a solution for all θ satisfying $-1 \leqq \theta \leqq 1$ if and only if the functions f_j and the (finite) sequence $\{\epsilon_{ij}\}$ possess the property (P).

G. Kreisel (Reading).

Rice, H. G. Recursive real numbers. Proc. Amer. Math. Soc. **5**, 784–791 (1954).

Specker [J. Symbolic Logic **14**, 145–158 (1949); these Rev. **11**, 151] mentioned four types (of definitions) of recursive (rec) real numbers: (A) rec sequences of rationals with a rec modulus of convergence, (B) rec decimal expansions, (C) rec Dedekind sections, (D) rec nested intervals. These classes are equivalent if 'rec' means general recursive [R. M. Robinson's review, ibid. **16**, 280–282 (1951), p. 282], but not if 'rec' means primitive recursive (Specker). [For comparison with intuitionist literature note (i) the meaning of equivalence, namely, in the extreme cases (A) and (C), from any given A-definition of r one gets a definition of a Dedekind section which is either general recursive or r is rational; since any rational has a C-definition, $C \supset A$, but the disjunction is, in general, undecided; (ii) if two rec reals are not equal there is an effective method of determining a positive lower bound of $|a-b|$ and therefore the distinction between $a \neq b$ and $a \# b$ of intuitionist literature does not arise.]

The author considers the class $\mathcal{S}$ of (general) rec (recursive) real numbers. By means of perfectly standard methods he shows that $\mathcal{S}$ is a field, $\mathcal{S}(i)$ is algebraically closed, and observes that $\mathcal{S}$ is not compact since there exists a bounded rec enumerable set of rationals which has no limit point in $\mathcal{S}$. He conjectures that if $0<a$, $a \in \mathcal{S}$, $b \in \mathcal{S}$, then $a^b \in \mathcal{S}$. Much more is known: (i) if $a_n \in \mathcal{S}$ and not all a_n are zero, the (real and imaginary parts of the) zeros of $\sum_{n=0}^{\infty} a_n z^n$ within its circle of convergence are in $\mathcal{S}$ [Kreisel, Nederl. Akad. Wetensch. Proc. Ser. A. **55**, 334–338 (1952); these Rev. **14**, 458]; (ii) $\mathcal{S}$ is not even conditionally compact since there exists a recursive (not merely recursively enumerable) bounded set of rationals which contains no rec subsequence with a rec modulus of convergence [Kreisel, J. Symbolic Logic **17**, 43–58 (1952), p. 57; these Rev. **14**, 440]. The standard proofs of the convergence of a^{b_n} to a^b, if $b_n \to b$, prove the author's conjecture. *G. Kreisel* (Reading).

***Sominski, I. S. Die Methode der vollständigen Induktion.** Deutscher Verlag der Wissenschaften, Berlin, 1954. 55 pp.

Translation of Sominskiĭ's "Metod matematičeskoĭ indukcii," 2d ed. [Gostehizdat, Moscow, 1952]. This is no. 3 in the series "Popular lectures on mathematics."

Borůvka, Otakar. Problems and paths of mathematics. Práce Moravskoslezské Akad. Věd Přírod., Brno **24**, 255–265 (1952). (Czech. Russian summary)

This paper contains a lecture given by the author in February 1952 at Brno. The following main points were discussed. 1. Proofs that mathematics belongs to the natural sciences and its relation to reality. 2. Internal and external forces for progress and development in mathematics. 3. The discovery and meaning of non-Euclidean geometry. 4. An example of the meaning of recent mathematical methods in the theory of classification. *From the author's summary.*

ALGEBRA

***Mostowski, Andrzej, i Stark, Marceli. Algebra wyższa. Część druga. [Higher algebra. Second part.]** Państwowe Wydawnictwo Naukowe, Warszawa, 1954. 173 pp. zł. 19.00.

[For the review of the first part see these Rev. **15**, 594.] The present volume contains the theory of polynomials with numerical coefficients, and together with the first volume contains all the material to be covered during the first year of the university. Abstract methods are avoided; elements of abstract algebra are to be given in volume three. Table of contents: Chapter IX, Polynomials in one variable; Ch. X, The rings of real and complex numbers (the fundamental theorem of algebra, various theorems on the estimates of the number of real roots); Chapter XI, Quadratic equations, equations of degree 3 and 4; Chapter XII, Polynomials in several variables, symmetric functions; Chapter XIII, The ring of rationals, algebraic and transcendental numbers; Chapter XIV, The theory of elimination. The volume is written with care and attention to didactic details, and contains a large number of problems, both computational and theoretical. *A. Zygmund* (Chicago, Ill.).

Majumdar, Kulendra N. On combinatorial arrangements. Proc. Amer. Math. Soc. **5**, 662–664 (1954).

The author's main result is the following theorem: "Let A, B be two matrices such that the product AB exists and the column sums of B and C are constant. Then if B is non singular the column sums of A are constant." The author obtains an interesting new result on block designs from this theorem by substituting for A an incidence matrix.

H. B. Mann (Columbus, Ohio).

Riordan, John. Discordant permutations. Scripta Math. **20**, 14–23 (1954).

Cette étude constitue un effort pour approcher de la solution du problème: énumérer les rectangles latins $n \times 4$. Soit une permutation des éléments $1, 2, \cdots, n$. Le $p$ème élément de cette permutation peut être égal à p, ou à $p+1$, ou à $p+2 \pmod n$; ou bien il peut n'être égal ni à p, ni à $p+1$, ni à $p+2 \pmod n$. Soit $N_{n,k}$ le nombre des permutations qui contiennent k éléments de la première sorte et $n-k$ de la seconde. La recherche de N est fondée sur la considération d'échiquiers particuliers successifs sur lesquels il s'agit de placer n tours, de manière qu'il n'y en ait jamais deux dans une même ligne ni dans une même colonne [cf. Kaplansky and Riordan, Duke Math. J. **13**, 259–268 (1943); ces Rev. **7**, 508]. Le cas d'un échiquier en "escalier" est d'abord examiné, puis celui d'un "escalier tronqué", enfin le cas relatif au problème lui-même. Représentant la fonction génératrice des N par:

$$N_n(y) = \sum_{k=0}^{n} N_{n,k} y^k = \sum_{0}^{n} \sigma_{n,k} (n-k)! (y-1)^k,$$

l'auteur donne, pour les σ, l'équation aux différences:

$$\sigma_{n,k} = \sigma_{n-1,k} + 2\sigma_{n-1,k-1} + \sigma_{n-2,k-1} - \sigma_{n-2,k-2} - 2\delta_{n-1,k},$$

où $\delta_{a,b}$ est égal à 1 ou à 0 suivant que a et b sont égaux ou distincts. Celle-ci permet, à défaut d'une expression indépendante, d'obtenir une valeur approchée des $N_{n,k}$ et d'en dresser la table pour les valeurs de n inférieures à 11.

A. Sade (Marseille).

García Pradillo, Julio. On properties and applications of permanents. Gaceta Mat. (1) 6, 8–14 (1954). (Spanish)

The applications mentioned in the title include computation of a closed formula for the sum of the products h at a time of n numbers, in particular the first n integers, e.g., $S_{n,3}=(n-1)n^2\cdot(n+1)^2(n-2)/48$. *J. L. Brenner.*

Carafa, Mario. Espressione e calcolo di un determinante di ordine n con un prodotto funzionale. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 11, 196–216 (1952).

"There is given a very simple explicit expression for the development of a determinant. From this it is shown that the value of a determinant of order n can be computed by a definite integral of a product of n polynomials, whose coefficients are the elements of the several columns of the given determinant" [from the author's summary]. The polynomials are of degrees up to about n^n, and are determined by solving certain diophantine problems. The author mentions the possibility of approximating the determinants by numerical quadrature of the integrals, but does not study the matter critically. Representations of other quantities as functional products were given by the author in an earlier paper [Collectanea Math. 1, no. 1, 1–62 (1948); these Rev. 10, 461]. *G. E. Forsythe* (Los Angeles, Calif.).

Horn, Alfred. Doubly stochastic matrices and the diagonal of a rotation matrix. Amer. J. Math. 76, 620–630 (1954).

A matrix $P=[p_{ij}]$ is said to be doubly stochastic if $p_{ij}\geqq 0$, $\sum_i p_{ij}=\sum_j p_{ij}=1$. If $x=(x_1, x_2, \cdots, x_n)$ is a vector in R^n, let $H(x)$ be the convex hull of the set of points of the form $(x_{\sigma(1)}, \cdots, x_{\sigma(n)})$, where σ runs over all permutations of $1, 2, \cdots, n$. The results of this paper may then be summarized as follows: (a) A real vector $y=(y_1, \cdots, y_n)$ can be the diagonal of a Hermitian matrix with eigenvalues $x_1, \cdots, x_n$ if and only if $y \in H(x)$; (b) a real vector y can be the diagonal of a rotation (proper orthogonal) matrix if and only if it belongs to the convex hull of the set of points of the form $(\pm 1, \pm 1, \cdots, \pm 1)$, an even number of the coordinates being equal to -1; (c) a real vector y can be the diagonal of a doubly stochastic matrix if and only if it belongs to the convex hull of the set of points of the form (ϵ_i), where ϵ_i is 0 or 1, and the number of values of i for which $\epsilon_i=0$ is not 1; (d) a real (complex) vector y can be the diagonal of an orthogonal (unitary) matrix if and only if the vector $(|y_1|, \cdots, |y_n|)$ satisfies condition (c). Other and equivalent formulations of these results are also given.

F. Smithies (Cambridge, England).

Gautschi, Werner. The asymptotic behaviour of powers of matrices. II. Duke Math. J. 20, 375–379 (1953).

[For part I see same J. 20, 127–140 (1953); these Rev. 15, 94.] By a general norm for the matrix A the author understands a functional $\Omega(A)$ which is positive $[\Omega(0)=0]$, homogeneous $[\Omega(\lambda A)=|\lambda|\Omega(A)]$, and which satisfies the triangle inequality. [See V. N. Fadeeva, Computational methods in linear algebra, Gostehizdat, Moscow-Leningrad, 1950, Chap. I; these Rev. 13, 872.] A bound for $\Omega(A^p)$ is given. Let $N_r(A)$ be the Hölder mean of the absolute values of the elements of A. Then $\lim_p [N_r(A^p)]^{1/p}$ exists and is equal to the absolute value of the dominant proper value (characteristic root) of A. *J. L. Brenner.*

Mihel'son, V. S. On the signs of the solution of a system of linear equations. Uspehi Matem. Nauk (N.S.) 9, no. 3(61), 163–170 (1954). (Russian)

This paper relates the problem of solving linear equalities or inequalities subject to restrictions as to signs of the solution to the problem of solving equalities which have matrices of maximum rank, with corresponding restrictions. The last-named equalities have fewer variables. The conditions given are not trivial, but seem complicated to apply.

J. L. Brenner (Aberdeen, Md.).

Sinden, Frank William. Ein Oszillationssatz für algebraische Eigenwertprobleme. Z. Angew. Math. Physik 5, 86–88 (1954).

A real matrix A is "variation-decreasing" [I. Schoenberg, Math. Z. 32, 321–328 (1930)] if, for every real vector x, the vector $y=Ax$ has no more sign changes than x. Suppose D is a diagonal matrix with all $d_{ii}>0$. Several results are announced: I. Suppose A is symmetric, positive definite, and variation-decreasing [or variation-increasing], with no zero codiagonal elements $a_{i,i+1}$, $a_{i,i-1}$. Then the eigenvalues of the problem $Ax=\mu Dx$ are $\mu_1>\mu_2>\cdots>\mu_n>0$, and the corresponding kth eigenvector has exactly $k-1$ [or $n-k$, respectively] sign changes. II. If $A=\|a_{ij}\|$ is variation-decreasing, then $A^*=\|(-1)^{i+j}a_{ij}\|$ is variation-increasing; A and A^* have identical spectra. III. Under the hypotheses of I, suppose u has exactly $k-1$ [or $n-k$, respectively] sign changes. Let $v=Au$. If $q_i=v_i(u_id_{ii})^{-1}>0$ (all i), then $\min(q_i)<\mu_k<\max(q_i)$. Result III, unlike the Collatz inclusion theorem [Eigenwertaufgaben mit technischen Anwendungen, Akademische Verlagsgesellschaft, Leipzig, 1949, p. 289; these Rev. 11, 137], states which eigenvalue is being localized. *G. E. Forsythe* (Los Angeles, Calif.).

Afriat, S. N. Composite matrices. Quart. J. Math., Oxford Ser. (2) 5, 81–98 (1954).

The results of this paper can be thought of as generalizations of the following. (i) The theorem on the characteristic roots, characteristic polynomial, and determinant of the Kronecker product of two matrices. (ii) A result of J. Williamson [Bull. Amer. Math. Soc. 37, 585–590 (1931)]. The author considers an $mn\times mn$ matrix A over the field F, partitions the matrix so that the submatrices are all $n\times n$ matrices A_{ij}, and assumes the A_{ij} to be commutative. In the algebra F_n of $n\times n$ matrices over F, A can be thought of as an $m\times m$ matrix, and as such as a determinant, since the subalgebra of F_n generated by the A_{ij} is commutative. This determinant is an element of F_n, and has in its turn a determinant in F. The value of this last determinant is shown to be the same as the determinant of the unpartitioned matrix A. Moreover, the characteristic roots of A are the characteristic roots of $(\alpha_{ij}^{(k)})$, where $\alpha_{ij}^{(k)}$ is the kth characteristic root of A_{ij} in some order. More detailed results appear, for example, concerning multiplicity. The author treats also the dual partitioning (A^{rs}) defined by the stipulation that the $(\bar{i}, \bar{j})$ element of A^{rs} is the (r, s) element of A_{ij}. *J. L. Brenner* (Aberdeen, Md.).

Brenner, J. L. A bound for a determinant with dominant main diagonal. Proc. Amer. Math. Soc. 5, 631–634 (1954).

The author recently improved Ostrowski's and Price's bounds for determinants with dominant main diagonal [C.

R. Acad. Sci. Paris **238**, 555–556 (1954); these Rev. **15**, 496]. A reduction of the given determinant to a determinant again with dominant main diagonal but smaller dimension was used. Even better bounds are obtained here by showing that the reduced determinant has a not less dominant diagonal than the original one. *O. Taussky-Todd.*

Bellman, Richard, and Hoffman, Alan. On a theorem of Ostrowski and Taussky. Arch. Math. **5**, 123–127 (1954).

The theorem of Ostrowski and Taussky referred to in the title states that if H_1 and H_2 are $n\times n$ hermitian matrices and H_1 is positive definite, then

$$|\det(H_1+iH_2)| \geqq \det H_1$$

[Nederl. Akad. Wetensch. Proc. Ser. A. **54**, 383–385 (1951); these Rev. **13**, 900]. The authors prove a result of Ky Fan that contains this as a special case [Amer. Math. Monthly **60**, 48 (1953)]; their proof uses the following lemma, which has some independent interest. Let $H=[h_{ij}]$ be an $n\times n$ hermitian matrix with eigenvalues $\alpha_1, \alpha_2, \cdots, \alpha_n$; write $\sigma(\alpha)$ for the vector $(\alpha_{\sigma(1)}, \cdots, \alpha_{\sigma(n)})$, where σ is any permutation of $1, 2, \cdots, n$, and let $K(\alpha)$ be the convex hull of the points whose position vectors are the $\sigma(\alpha)$. Then the point $\eta=(h_{ii})$ belongs to $K(\alpha)$. Some other related results are also obtained. *F. Smithies* (Cambridge, England).

Richter, Hans. Bemerkung zur Norm der Inversen einer Matrix. Arch. Math. **5**, 447–448 (1954).

Let A be an $n\times n$ matrix (a_{ij}); let $N(A)=\sum|a_{ij}|^2$. Let Q be $(\det A)A^{-1}$. Then $N(Q)$ satisfies the inequality $N(Q)\leqq(1/n)^{\frac{1}{2}(n-2)}[N(A)]^{n-1}$, with equality only when A^*A is a scalar matrix. If $\det A$ is 0, the inequality still holds, with Q now the adjoint of A. *J. L. Brenner.*

Banachiewicz, T. Sur les thèses de deux notes de T. Ważewski relatives aux cracoviens et matrices. Ann. Soc. Polon. Math. **24** (1951), no. 2, 153–156 (1954).

The author defends his algorithm [for example, Bull. Int. Acad. Polon. Sci. Cl. Sci. Math. Nat. Ser. A. Sci. Math. **1938**, 134–135] for solving a system of linear algebraic equations against criticisms attributed to Ważewski [Ann. Soc. Polon. Math. **22**, 286–287, 288 (1950); which the reviewer has not seen]. The author's major plea is for the use of cracovians (which have column-by-column multiplication) in the literature instead of matrices, on grounds of theoretical equivalence and greater practical convenience. [For Ważewski's reply, see the following review.] *G. E. Forsythe* (Los Angeles, Calif.).

Ważewski, T. Sur l'algorithmisation des méthodes d'éliminations successives. Ann. Soc. Polon. Math. **24** (1951), no. 2, 157–164 (1954).

The author carefully replies to Banachiewicz's article [see the preceding review], which he calls a polemic. The reviewer finds Ważewski the more objective of the two, but the issues are of too little importance to report here. *G. E. Forsythe* (Los Angeles, Calif.).

Murnaghan, Francis D. On the analyses of $\{m\}\otimes\{1^k\}$ and $\{m\}\otimes\{k\}$. Proc. Nat. Acad. Sci. U. S. A. **40**, 721–723 (1954).

In a previous note [same Proc. **37**, 439–441 (1951); these Rev. **13**, 201] the author has shown that the terms in the expansions of $\{m\}\otimes\{1^k\}$ and $\{m\}\otimes\{k\}$ which correspond to exactly k parts may be obtained from the expansions of $\{m-1\}\otimes\{k\}$ and $\{m-1\}\otimes\{1^k\}$ by the formulae

$$(\{m\}\otimes\{1^k\})'=\{m-1\}\otimes\{k\},\quad(\{m\}\otimes\{k\})'=\{m-1\}\otimes\{1^k\},$$

where the prime indicates that every part is decreased by unity, partitions into less than k parts being omitted. The method is now extended to give the partitions into exactly $(k-j)$ parts by the formulae

$$\begin{aligned}(\{m\}\otimes\{1^k\})'=&\{m\}(\{m\}\otimes\{1^{k-1}\})'\\&-(\{m\}\otimes\{2\})(\{m\}\otimes\{1^{k-2}\})'+\cdots\\&+(-1)^{j-1}(\{m\}\otimes\{j\})(\{m\}\otimes\{1^{k-j}\})'\\&+(-1)^j(\{m-1\}\otimes\{k\})\{j\}\\(\{m\}\otimes\{k\})'=&\{m\}(\{m\}\otimes\{k-1\})'\\&-(\{m\}\otimes\{1^2\})(\{m\}\otimes\{k-2\})'+\cdots\\&+(-1)^{j-1}(\{m\}\otimes\{1^j\})(\{m\}\otimes\{k-j\})'\\&+(-1)^j(\{m-1\}\otimes\{1^k\})\{j\},\end{aligned}$$

the prime denoting that only partitions into exactly $(k-j)$ parts are taken and unity is subtracted from each part. These lead to a complete analysis of $\{m\}\otimes\{1^k\}$ and $\{m\}\otimes\{k\}$. Examples are given. *D. E. Littlewood* (Bangor, Wales).

Abstract Algebra

Croisot, Robert. Quelques applications et propriétés des treillis semi-modulaires de longueur infinie. Ann. Fac. Sci. Univ. Toulouse (4) **16** (1952), 11–74 (1953).

This paper together with an earlier one [Ann. Sci. Ecole Norm. Sup. (3) **68**, 203–265 (1951); these Rev. **13**, 718] comprises the author's thesis [Paris, 1951]. Many of the results have been announced earlier by the author [C. R. Acad. Sci. Paris **226**, 616–617, 767–768 (1948); **228**, 1090–1092 (1949); **231**, 95–97, 1399–1401 (1950); **232**, 27–29 (1951); these Rev. **9**, 406; **10**, 508; **12**, 5, 473; **13**, 6]. Modular lattices, both in general and with a chain condition, are characterized in terms of conditions from the first half. Let T be a complete, $\cap$-continuous lattice in which $x_1\cap x_2=0$, $(x_1\cup x_2)\cap x_3=0$, and x_3 covers 0 together imply $(x_1\cup x_3)\cap x_2=0$. In T, if I is the union of points, then every element has a complement; if every element is the union of points then every element of the sublattice $[a, b]$ has a complement in that sublattice. Each of the conditions for semi-modularity studied in the first half is preserved in passing to convex sublattices, but not to general sublattices, or to $\cap$-, $\cup$-, or lattice-homomorphisms, contrary to G. Birkhoff [Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948, p. 103; these Rev. **10**, 673]; most hold in passing to cardinal product. The question of equality of length of chains with the same ends is studied, with use of Zassenhaus refinements. Finally, partial hypergroups (associative systems with a subset of scalar units, in which each element has a unique right inverse and unique left inverse) are axiomatized. The result on equality of chains is applied to such systems, yielding as special cases known results on chains of subgroups and of equivalence relations. *P. M. Whitman.*

Dilworth, R. P. Proof of a conjecture on finite modular lattices. Ann. of Math. (2) **60**, 359–364 (1954).

Using the Moebius-Weisner function [L. Weisner, Trans. Amer. Math. Soc. **38**, 474–484 (1935)], the author proves the following important theorem for any finite modular lattice L. If V_k denotes the set of elements of L covered by precisely k elements, and W_k denotes the set of elements of

L covering precisely k elements, then V_k and W_k contain the same number of elements. In particular, the number of meet irreducibles of L equals the number of join irreducibles. [See Schützenberger, C. R. Acad. Sci. Paris **218**, 818–819 (1944), whose discussion is incomplete; these Rev. **7**, 409.] *G. Birkhoff.*

Barbilian, D. Normalités localement ou intégralement involutives. Acad. Repub. Pop. Române. Stud. Cerc. Mat. **4**, 29–67 (1953). (Romanian. Russian and French summaries)

The paper concerns generalizations of the Jordan-Hölder ("Schreier-Kurosh-Zassenhaus") theorem, based on "normality" conditions generalizing those of Uzkov [Mat. Sbornik N.S. **4**(**46**), 31–43 (1938)] and Kořínek [Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. **1941**, no. 14 (1942); these Rev. **7**, 509; **13**, 1138]. Various applications are given to transfinite chains, similar to those given by Kurosh [Mat. Sbornik N.S. **16**(**58**), 59–72 (1945); these Rev. **8**, 253]. *G. Birkhoff.*

Povarov, G. N. On functional separability of Boolean functions. Doklady Akad. Nauk SSSR (N.S.) **94**, 801–803 (1954). (Russian)

A method is given for determining whether or not a Boolean function is functionally separable in the sense of Shannon [Bell System Tech. J. **28**, 59–98 (1949); these Rev. **10**, 671]. This theorem is applied to symmetric functions to show that the only symmetric functions which are functionally separable are linear Boolean sums or linear sums modulo two. *C. Saltzer* (Cleveland, Ohio).

Choudhury, A. C. On Boolean narings. Bull. Calcutta Math. Soc. **46**, 41–45 (1954).

The author's main result is that an alternative ring in which every element is idempotent is commutative and associative. This is a special case of a result due to Smiley [Bull. Amer. Math. Soc. **53**, 775–778 (1947); these Rev. **9**, 172]. *I. Kaplansky* (Chicago, Ill.).

Fujiwara, Tsuyoshi. On the structure of algebraic systems. Proc. Japan Acad. **30**, 74–79 (1954).

This paper is closely associated with the studies of Shoda on generalized algebraic systems [see the following review]. The author introduces new definitions of co-sets and normality for subsystems and on the basis of these he proceeds to give extensions of the Jordan-Hölder-Schreier theorem for chains and the Remak-Schmidt-Ore theorem for direct decompositions. *O. Ore* (New Haven, Conn.).

Shoda, Kenjiro. Über die nicht algebraischen Erweiterungen algebraischer systeme. Proc. Japan Acad. **30**, 70–73 (1954).

The author continues his studies of general algebraic theories [General algebra, Kyôritsu-shuppan, Tokyo, 1947; Osaka Math. J. **1**, 182–225 (1949); **4**, 133–143 (1952); these Rev. **12**, 313; **11**, 308; **14**, 614]. In the present paper some of the results on transcendental adjunctions are extended to this wider kind of algebraic systems. *O. Ore.*

Nagata, Masayoshi. Some remarks on local rings. II. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **28**, 109–120 (1954).

[Pour parti I voir Nagoya Math. J. **6**, 53–58 (1953); ces Rev. **15**, 391.] Rappelons qu'un anneau local $\mathfrak{o}$ d'idéal maximal $\mathfrak{m}$ est dit henselien [Nagata, ibid. **5**, 45–57 (1953); ces Rev. **14**, 529] s'il est d'intégrité et si toute décomposition mod $\mathfrak{m}$ d'un polynome $f(X)$ sur $\mathfrak{o}$ se prolonge en une décomposition de $f(X)$ sur $\mathfrak{o}$. L'auteur montre d'abord qu'un anneau local régulier henselien est algébriquement fermé dans son complété $\hat{\mathfrak{o}}$, pourvu que tous ses quotients $\mathfrak{o}/\mathfrak{p}$ ($\mathfrak{p}$: idéal premier minimal) jouissent de la même propriété. Ainsi, si l'on dispose d'une classe G d'anneaux locaux réguliers et henseliens (dite "classe de Weierstrass") telle que, pour tout $\mathfrak{o} \in G$ et pour tout idéal premier minimal $\mathfrak{p}$ de $\mathfrak{o}$, $\mathfrak{o}/\mathfrak{p}$ soit un module de type fini sur un anneau de la classe G, alors: a) tout élément $\mathfrak{o}$ de G est algébriquement fermé dans son complété; b) tout anneau local d'intégrité $\mathfrak{r}$ qui est un module de type fini sur $\mathfrak{o} \in G$ est analytiquement irréductible, et $\hat{\mathfrak{r}}$ est intégralement clos quand $\mathfrak{r}$ est intégralement clos; c) l'idéal $\mathfrak{p}\cdot\hat{\mathfrak{o}}$ est premier pour tout idéal premier $\mathfrak{p}$ de $\mathfrak{o} \in G$. Les anneaux de séries convergentes forment une classe de Weierstrass; l'auteur donne d'ailleurs une démonstration directe du fait que l'anneau des séries convergentes à n variables est algébriquement fermé dans l'anneau des séries formelles à n variables. Le résultat b) ci dessus, et la théorie des "henselisations" d'anneaux locaux (loc. cit.) permettent de généraliser à des classes d'anneaux plus vastes que celle des anneaux locaux géométriques les théorèmes d'irréductibilité et de normalité analytiques de Zariski [Ann. Inst. Fourier Grenoble **2**, 161–164 (1951); ces Rev. **13**, 579]. L'une de ces classes contient les anneaux locaux "d'inégales caractéristiques" qui se rencontreront dans la "Géométrie Algébrique sur les corps de nombres". Dans toutes ces généralisations l'existence de "noyaux" (de types, à vrai dire, plus variés que ceux de Chevalley-Zariski) est mise dans les hypothèses.

La lecture de ce mémoire est rendue difficile par de constantes références à des articles non encore publiés. Les démonstrations faciles sont laissées au lecteur, ce qui serait tout à fait admissible si celui ci était prévenu des passages qu'il lui faut restituer. Or ce n'est pas toujours le cas. Par exemple l'énoncé d'un théorème annonce que l'anneau des séries convergentes $\mathfrak{o}$ est algébriquement fermé dans l'anneau $\hat{\mathfrak{o}}$ des séries formelles, mais la démonstration montre seulement que $\mathfrak{o}$ est intégralement fermé dans $\hat{\mathfrak{o}}$, sans un mot pour prévenir le lecteur que le passage à la conclusion annoncée est facile (mais non trivial!) à partir de là. *P. Samuel.*

Nagata, Masayoshi. Note on integral closures of Noetherian domains. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **28**, 121–124 (1954).

Exemples d'anneaux locaux noethériens et sans diviseurs de zéro A et B tels que a) un anneau compris entre A et sa clôture intégrale A' n'est pas noethérien; b) la clôture intégrale B' de B n'est pas un anneau noethérien. On part d'un corps K de caractéristique p tel que $[K:K^p]$ soit infini, et des anneaux $\mathfrak{o}_n = K^p[[X_1, \cdots, X_n]][K]$ et $\mathfrak{r}_n = K[[X_1, \cdots, X_n]]$. L'anneau $\mathfrak{o}_n$ est un anneau local régulier, et $\mathfrak{r}_n$ est son complété. On prend pour A un anneau intermédiaire entre $\mathfrak{o}_2$ et $\mathfrak{r}_2$, et pour B un anneau intermédiaire entre $\mathfrak{o}_3$ et $\mathfrak{r}_3$. *P. Samuel* (Clermont-Ferrand).

Hinohara, Yukitoshi. Shoda's condition on quasi-Frobenius rings. Proc. Japan Acad. **30**, 291–293 (1954).

A ring A is said to satisfy Shoda's condition for left ideals if A has a left unit element and every A-homomorphism between two left ideals is given by the right multiplication by an element of A. If this result is assumed only for homomorphisms onto simple left ideals we speak of the modified Shoda's condition. The author shows that A is a quasi-

Frobenius ring, if and only if the modified condition is satisfied. This also shows that the modified Shoda's condition implies Shoda's condition. *R. M. Thrall.*

Yoshii, Tensho. Note on generalised uniserial algebras. I. Osaka Math. J. **6**, 105–107 (1954).

This paper extends earlier results of G. Azumaya and T. Nakayama [Jap. J. Math. **19**, no. 4, 263–273 (1948); these Rev. **11**, 316, 872] and K. Morita [Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. **4**, 177–194 (1951); these Rev. **14**, 246]. An associative algebra A with unity element is said to be generalized uniserial if each primitive left ideal and each primitive right ideal possesses only one composition series; it is said to be uniserial if, moreover, each primitive left ideal has all of its compositions factors isomorphic. We say that A is absolutely generalized uniserial if it remains generalized uniserial after every extension of the coefficient field. The main result of this note is that an algebra A is absolutely generalized uniserial if and only if it is a direct sum A_1+A_2, where A_2 is absolutely uniserial and A_1 is a generalized uniserial algebra whose residue class algebra modula its radical is separable. *R. M. Thrall.*

Freudenthal, Hans. Beziehungen der E_7 und E_8 zur Oktavenebene. I. Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indagationes Math. **16**, 218–230 (1954).

This paper is a continuation of a sequence of papers [same Proc. **56**, 81–89, 90–94, 95–98 (1953); these Rev. **14**, 948]. Let C be the Cayley division ring of order 8 over the reals. Then the author considers the 3 by 3 matrices hermitian over C. These form a ring $\mathfrak{J}$ closed under a Jordan product, of order 27 over the reals. In terms of the Jordan product $A \circ B$ and of a formally defined determinant several other operations are defined, and numerous relationships are found. This paper deals with the 56-dimensional space K with components X, U, ξ, ω with X and U from $\mathfrak{J}$ and ξ, ω real numbers. The projective transformations of K which leave a 27-dimensional maniform M fixed yield a representation of the exceptional Lie group E_7.

Marshall Hall Jr. (Columbus, Ohio).

Zassenhaus, Hans. The representations of Lie algebras of prime characteristic. Proc. Glasgow Math. Assoc. **2**, 1–36 (1954).

With the ultimate aim of dissecting and cataloging the representations of finite-dimensional Lie algebras of characteristic p or suitable classes thereof, the author puts to use methods designed to give precise interpretation, in the universal associative algebra $A(L)$ of the Lie algebra L, to such problems as those connected with the indecomposable and irreducible representations and the nature of the irreducible constituents of an indecomposable representation. For example, if L is a Lie algebra over a field F of characteristic p, the center Z of $A(L)$ is a normal algebraic variety over F, specializations of Z onto primary rings over F correspond to indecomposable representations of L, specializations of Z onto F correspond to absolutely irreducible ones, and the irreducible components of an indecomposable representation belong to the same specialization of Z. [Much of this work appeared in the author's lectures at the First Summer Mathematical Institute, a summary of which may be found in Bull. Amer. Math. Soc. **60**, 463–470 (1954).]

W. G. Lister (Providence, R. I.).

Iwahori, Nagayosi. On some matrix operators. J. Math. Soc. Japan **6**, 76–105 (1954).

Let $\mathfrak{g}$ be a Lie algebra over a field K and $\mathfrak{R}$ the set of all representations of $\mathfrak{g}$. By a representation of $\mathfrak{R}$ is meant a mapping which assigns to every element of $\mathfrak{R}$ a square matrix of the same degree as this element, and which behaves cogrediently with respect to the operations on $\mathfrak{R}$ (similarity transformations, Cartesian sums, tensor sums). The representations of $\mathfrak{R}$ form a Lie algebra $\mathfrak{g}^*$ which contains $\mathfrak{g}$. By a theorem of Harish-Chandra [Ann. of Math. (2) **51**, 299–330 (1950); these Rev. **11**, 492], $\mathfrak{g}^*=\mathfrak{g}$ when $\mathfrak{g}$ is semi-simple and K algebraically closed of characteristic 0. The author considers the case where $\mathfrak{g}$ is 1-dimensional and K algebraically closed of any characteristic; $\mathfrak{g}^*$ then turns out to be always infinite-dimensional. In that case, $\mathfrak{R}$ may be identified to the set of all matrices (of all degrees), and the author calls the representations of $\mathfrak{R}$ sum-sum (s-s) operators. Besides the s-s operators, the author also determines completely the s-p operators, which are defined like the s-s operators except that they transform tensor sums into tensor products, p-s operators which change tensor products into tensor sums, and p-p operators which change tensor products into tensor products. The p-s operators turn out to be trivial, but the author determines the p-s operators defined only on the set $\mathfrak{S}$ of regular matrices. In all cases, the problem is reduced to the separate consideration of the effect of an operator ζ on the diagonal matrices and on the nilpotent matrices N (or the unipotent matrices $I+N$, I the unit matrix, if ζ is p-s or p-p). The effect on diagonal matrices is determined by the effect on matrices of degree 1, i.e. by a mapping g of K (or of the set K^* of elements $\neq 0$ in K) into K; g must change sums into sums if ζ is s-s, sums into products if ζ is s-p, etc. If K is of characteristic $\neq 0$, then the only s-p or p-s operators are trivial; if not, they operate on nilpotent (or unipotent) matrices by means of the exponential or the logarithmic mapping. If K is of characteristic 0, an s-s operator ζ transforms a nilpotent matrix N into cN, c a constant; if K is of characteristic p, then $\zeta(N)=\sum_0^\infty c_i N^{p^i}$, (c_i) being an arbitrary sequence of constants. Let now ζ be a p-p operator. If ζ is nontrivial and K of characteristic 0, then, for a unipotent matrix N, $\zeta(I+N)=\exp c(\log(I+N))$, c a constant, while, if K is of characteristic $p>0$, $\zeta(I+N)=\prod_0^\infty (I+N^{p^i})^{c_i}$, the c_i's being fixed integers.

It follows that a matrix B is a replica of a matrix A if and only if $B=\zeta(A)$ with some s-s operator ζ. The author then generalizes the notion of replica to introduce s-p replicas, p-s replicas and p-p replicas (in each case, one has only to change tensor sums into tensor products and to replace $-{}^tA$ by ${}^tA^{-1}$ at suitable places in the definition, and to limit oneself to regular matrices when it is necessary for the definition to make sense). In all cases (but for one trivial exception) it remains true that B is a replica of A if and only if $B=\zeta(A)$ with some operator ζ of the suitable type. In particular, if K is of characteristic $p>0$, then the only matrix which admits as invariants (in the ordinary sense) all infinitesimal invariants of a nilpotent matrix is the identity, and the matrices which admit as invariants all invariants of a unipotent matrix form a finite group. This gives, in the reviewer's opinion, a very clear explanation of the fact that an algebraic group is not uniquely determined by its Lie algebra in the case of characteristic p.

The results on operators of various kinds are partially extended to cover the case of a basic field which is not algebraically closed. The extension is complete if one limits

oneself to the consideration of diagonal and nilpotent (or unipotent) matrices, but not for the mixed cases.

C. Chevalley (New York, N. Y.).

Hochschild, G. Cohomology of restricted Lie algebras. Amer. J. Math. 76, 555–580 (1954).

In accordance with the principle that every kind of algebraic structure should have its own cohomology theory, the author develops the cohomology theory for the structure of a restricted Lie algebra over a field of characteristic p. The cohomology theory of an ordinary Lie algebra L may be defined in two different manners: either by using the Lie cochains, which are alternating multilinear functions on L, or by using associative cochains, which are multilinear functions on the enveloping algebra of L. It is the second approach which the author generalizes to restricted algebra, by simply replacing the enveloping algebra R of a restricted algebra L by its u-algebra, which is deduced from R by adjoining the new relations $x^p - x^{[p]} = 0$, where $x \to x^{[p]}$ is the supplementary operation which is given in the structure of L. Proceeding in this manner, it is not difficult to define the cohomology groups $H_*^m(L, M)$, where M is an L-module (i.e. the space of a restricted representation of L), but it is much more difficult than in the classical case to derive the classical interpretations of the first and second cohomology groups: to do this is one of the objects of the present paper. On the other hand, the u-algebra being a factor of the enveloping algebra, there is a canonical homomorphism of $H_*^m(L, M)$ into $H^m(L, M)$, which is the mth cohomology group of the underlying Lie algebra of L in M; the second object of the paper is to study this homomorphism in the cases $m = 1, 2$.

Inasmuch as the interpretation of the groups $H_*^m(L, M)$ is concerned, one arrives at exactly the same results as in the classical case. First, let K and Q be L-modules, and M the L-module of all linear maps of Q into K. A structure of "extension of K by Q" is by definition constituted by a monomorphism j of K into an L-module E and by an epimorphism of E on Q whose kernel is $j(K)$. Each type of structure carrying automatically its notion of isomorphism, we may speak of the set of types of extensions of K by Q; this set is in a 1-1 correspondence with $H_*^1(L, M)$, which gives the classical interpretation of the latter group. Now, let there be given an abelian restricted algebra M which has at the same time a structure of L-module and is such that $M^{[p]}$ is annihilated by L. By an extension of M by L is meant a structure constituted by a monomorphism j of M into a restricted algebra E and an epimorphism h of E on L of kernel $j(M)$ which is such that the L-module structure which may be defined on M by means of its imbedding j in E and of the homomorphism h of E on L coincides with the one given a priori. Then the author shows that the set of types of extensions of M by L has a structure of additive group $\mathrm{ext}_*(M, L)$, and even of vector space (over the basic field of L) if M is strongly abelian (i.e. if $M^{[p]} = \{0\}$); in the latter case, $\mathrm{ext}_*(M, L)$ is canonically isomorphic to $H_*^2(L, M)$.

It turns out that the canonical homomorphism

$$H_*^1(L, M) \to H^1(L, M)$$

is a monomorphism; the author gives a characterization of its image group. In the same notation as above, there is also a canonical homomorphism of the group $\mathrm{ext}_*(L, M)$ into $H^2(L, M)$, which reduces to the homomorphism

$$H_*^2(L, M) \to H^2(L, M)$$

when M is strongly abelian. In general, the image group of $\mathrm{ext}_*(L, M)$ is composed of those elements of $H^2(L, M)$ which satisfy the following condition: if E is an extension of the ordinary Lie algebra M by the ordinary Lie algebra L corresponding to an element of this image group, then E can be given a restricted structure such that its homomorphism on L is restricted. The author also determines the kernel of the mapping $\mathrm{ext}_*(L, M) \to H^2(L, M)$ by means of an exact sequence which it would be a little too long to describe here.

In the last section the author states that the problem of classifying the extensions by L of an arbitrary restricted algebra M on which L "operates" by means of a homomorphism of L into the algebra of outer derivations of M (i.e. the quotient of the algebra of derivations by the ideal of inner derivations) may be reduced to the same problem for strongly abelian algebras M (analogously to what happens in group theory); this is expressed by saying that a certain mapping is a bijection, but the author restricts himself to the proof that it is a surjection. *C. Chevalley.*

Hochschild, G. Lie algebra kernels and cohomology. Amer. J. Math. 76, 698–716 (1954).

In this paper, the author gives an interpretation for the third cohomology group $H_*^3(L, N)$ of a restricted Lie algebra L in an L-module N (cf. preceding review for the definitions of the terms). This interpretation, which is new even for ordinary Lie algebras, is analogous to the interpretation of the third cohomology group in group theory. By an L-kernel M is meant a structure constituted by a restricted Lie algebra M and a homomorphism of L into the algebra of outer derivations of M; the center of M then has a structure of L-module, and this module, given with its structure of restricted algebra, is called the nucleus of the L-kernel. The set Σ of types of L-kernels with a given type of nucleus has an associative and commutative law of composition, under which however it does not constitute a group. One obtains a group by going over to the quotient of Σ by an equivalence relation R defined as follows. Any extension of M by L defines on M a structure of L-kernel; an L-kernel which may be obtained in this manner is called extendible. This being said, two L-kernels M, M' are considered as equivalent if there are extendible kernels U, V such that $(M)+(U)=(M')+(V)$ (where (M) is the type of M). If we assume that the center N of M is strongly abelian, then it turns out that the set of equivalence classes of types of L-kernels with nucleus N has a structure of vector space over the basic field L of F; and the author proves that this vector space is isomorphic to $H_*^3(L, M)$. The proof necessitates some very elaborate constructions and many computations, of which a good number are left to the reader to perform alone and unaided. In the formula which is written on l. 14 from bottom, p. 708, one should read $x_1 \cdot q_2 - x_2 \cdot q_1$ instead of $x_1 \cdot q_1 - x_2 \cdot q_2$. At various places on pp. 709–710–711, the author uses the notation abc, where a, b, c are elements of a non-associative algebra Q; one should read $(ab)c$; moreover, the product ab may be interpreted indifferently as a product in U_L or in Q, because, in all cases where the author uses this notation, the two have the same value.

C. Chevalley (New York, N. Y.).

Theory of Groups

Tamari, Dov. Monoïdes préordonnés et chaînes de Malcev. Bull. Soc. Math. France 82, 53–96 (1954).

Reviewed earlier as the author's thesis [Paris, 1951; these Rev. 14, 532].

Inaba, Eizi. Zur Bildung der auflösbaren Gruppen. J. Math. Soc. Japan 6, 106–113 (1954).

Let G_0 be a solvable group of finite order, let l be a prime, let H be a normal abelian subgroup of order l^m and type $(l, \cdots, l)$, and let $G=G_0/H$. Then G_0 is called an l-expansion (l-Aufspaltung) of G. We may regard H as a vector space of dimension m over the prime field P of characteristic p. If g_sH is the coset of G_0 whose image in G is s, then the automorphism $\Lambda(s)$: $h \to s(h)=g_shg_s^{-1}$ of H is well defined for each s in G and the set of all $\Lambda(s)$ forms a representation Λ of G. We then say that G_0 is an l-expansion of G with associated representation Λ. Let S be an l-Sylow subgroup of G, and let Λ_0 be the representation of G induced by the identity representation of S. An l-expansion of G is called regular if its associated representation is Λ_0. The author shows that for every l-expansion $\bar{G}=(H, G)$ there exists a contraction $G_1=(H_1, G)$ of a regular l-expansion of G and a G admissible subgroup H_2 of H_1 such that the factor group G_1/H_2 and $\bar{G}$ are equivalent l-expansions of G.

A finite group G is said to be regular solvable if it has a composition series $G=G_0 \supset G_1 \supset \cdots \supset G_\lambda=\{e\}$ for which G/G_{i+1} is a regular l_i-expansion of G/G_i, where l_i is a prime divisor of the order of G. The author proves that every solvable group is a homomorphic image of a regular solvable group.

R. M. Thrall (Ann Arbor, Mich.).

Cockcroft, W. H. Interpretation of vector cohomology groups. Amer. J. Math. 76, 599–619 (1954).

The first result of this paper employs the vector cohomology groups of J. H. C. Whitehead [Quart. J. Math., Oxford Ser. (2) 1, 219–228 (1950); these Rev. 12, 156] to generalize a theorem of Baer [Automorphismen von Erweiterungsgruppen, Hermann, Paris, 1935] on the automorphisms of a group extension. Let groups $E/G=Y$ with operators from X, and $E'/G'=Y'$ with operators from X', be given, where G and G' are abelian. Let matching homomorphisms $X \to X'$, $G \to G'$, $Y \to Y'$ be given. Then the set of matching homomorphisms α: $E \to E'$ corresponds one-to-one with the vector cohomology group $\mathfrak{Z}^1(X, Y; G')$, with cohomologous elements corresponding to α, α' that differ by the inner automorphism induced by some element of G' invariant under X'. When the two given extensions coincide, this one-to-one correspondence becomes a group isomorphism.

The second result is an extension of the theorems of Eilenberg and MacLane [Duke Math. J. 14, 435–463 (1947); these Rev. 9, 132] on the classification of loop prolongations. For loops with operators (which are not required to effect endomorphisms), the analogous theorems hold, employing the higher vector cohomology groups. The proof is reduced to the simple case by the device of Whitehead of replacing Y by its split extension by X; a direct proof is indicated.

R. C. Lyndon (Ann Arbor, Mich.).

Parker, E. T. A simple group having no multiply transitive representation. Proc. Amer. Math. Soc. 5, 606–611 (1954).

The question whether every simple group possesses a multiply transitive permutation representation is answered in the negative by showing that the group of 4×4 matrices with coefficients in GF[2^2], determinant unity, and which permute among themselves the vectors of order 4 having 4 non-zero or 2 zero and 2 non-zero marks of the field, form a simple group of order $2^6 \cdot 3^4 \cdot 5$ which is isomorphic with no multiply transitive permutation group. The author shows that if the group has a permutation representation which is at least doubly transitive, of degree n, then $n! \geqq 2^6 \cdot 3^4 \cdot 5$, and $n(n-1)$ is a divisor of $2^6 \cdot 3^4 \cdot 5$. Thus n must be 9, 10, 16 or 81. These 4 possible values of n are examined separately and by arguments related to Sylow subgroups, normalisors and commutator subgroups each is shown to be inadmissible. A referee has remarked that a shorter proof of the non-existence of a multiply transitive representation could be obtained by the use of Frame's table of characters of the group [Duke Math. J. 2, 477–484 (1936)].

D. E. Littlewood (Bangor, Wales).

Ehrenfeucht, A., et Łoś, J. Sur les produits cartésiens des groupes cycliques infinis. Bull. Acad. Polon. Sci. Cl. III. 2, 261–263 (1954).

Let C^T be the additive group of integer-valued functions on T, C^{T^*} the subgroup generated by the characteristic functions of single elements of T. If the cardinality of T is less than the first inaccessible alef, it is known that two homomorphisms of C^T into the group C of integers are necessarily equal if their restrictions to C^{T^*} are equal. Assuming T to be such that this proportion is true, the author announces theorems about the form of homomorphisms $C^T \to C$, about the endomorphisms of C^T and about the group Hom (C^T, C).

P. A. Smith (New York, N. Y.).

Itô, Noboru. On monomial representations of finite groups. Osaka Math. J. 6, 119–127 (1954).

In the first part of this paper the author gives new proofs of Shoda's results on monomial representations of finite groups [Proc. Phys.-Math. Soc. Japan (3) 15, 249–257 (1933)] and also gives some remarks on these results. In the second part of the paper he proves that if an irreducible monomial representation of a finite group is induced by a proper cyclic subgroup then at least one group element is represented by a diagonal but not scalar matrix.

R. M. Thrall (Ann Arbor, Mich.).

Wolf, Paul. Der Dualitätssatz der Darstellungstheorie endlicher Gruppen als Aussage über den Gruppenring. Math. Nachr. 11, 129–133 (1954).

Let $\mathfrak{G}$ be a finite group and let K be a field whose characteristic does not divide the order of $\mathfrak{G}$ and over which irreducible representations of $\mathfrak{G}$ are absolutely irreducible. Let A_i: $S \to A_i(S)$ $(i=1, \cdots, k)$ be the irreducible representations of $\mathfrak{G}$ and let X_i be the character of A_1. Let

$$X_iX_j=\sum C_{ij}^\nu X_\nu.$$

Then there exist matrices P_{ij} $(i, j=1, \cdots, k)$ such that $P_{ij}^{-1}(A_i \times A_j)P_{ij}$ is the diagonal form $\sum_\nu C_{ij}^\nu A_\nu$. Let A_i have degree f_i. Suppose (i) that M_i is a matrix of degree f_i $(i=1, \cdots, k)$, (ii) that not all M_i are zero, and (iii) that $P_{ij}^{-1}(M_i \times M_j)P_{ij}=\sum_\nu C_{ij}^\nu M_\nu$ $(i, j=1, \cdots, k)$. Then there exists S in $\mathfrak{G}$ such that $M_i=A_i(S)$ $(i=1, \cdots, k)$.

R. M. Thrall (Ann Arbor, Mich.).

Robinson, G. de B., and Taulbee, O. E. The reduction of the inner product of two irreducible representations of S_n. Proc. Nat. Acad. Sci. U. S. A. 40, 723–726 (1954).

If a group G of order g has a subgroup H of order h, then a matrix representation (γ) of H, of degree f_γ, induces a

representation of G of degree gf_γ/h, which is denoted by $(\gamma)H \uparrow G$. Also a representation (β) of G gives immediately, by restricting the elements represented to the subgroup H, a representation of H which is denoted by $(\beta)G \downarrow H$.

If (I) denotes the identity representation and $\times$ the Kronecker product of two representations, then the authors show that $((I)H \uparrow G) \times (\beta) = ((\beta)G \downarrow H) \uparrow G$. Taking G to be S_n, the symmetric group of degree n, and H_α the subgroup $S_{\alpha_1} \times S_{\alpha_2} \times \cdots \times S_{\alpha_k}$, where $(\alpha_1\alpha_2 \cdots \alpha_k)$ is a partition of n, the theorem leads to a method of determining $((I)H_\alpha \uparrow S_n) \times (\beta)$ from which the expansion of $(\alpha) \times (\beta)$ can be determined. The method of calculation, though an improvement on Murnaghan's method [Amer. J. Math. 60, 761–784 (1938)], would nevertheless leave room to think that further simplification was possible. *D. E. Littlewood* (Bangor, Wales).

Murnaghan, Francis D. On the symmetry properties of powers of representations of the two-dimensional unimodular unitary group. Proc. Nat. Acad. Sci. U. S. A. 40, 822–825 (1954).

The author remarks on the importance of the analysis of the powers of representations of the two-dimensional unimodular unitary group, since the results give an equivalent analysis of the 3-dimensional rotational group. His analysis depends on the reduction formula

$$\{m\} \otimes k = (\{m-2\} \otimes k) + (\{m-1\} \otimes k-1)(\{m+k-1\} - \{m+k-3\}) + \{m\} \otimes k-2,$$

together with the formula $\{m\} \otimes 1^k = \{m-k+1\} \otimes k$. These formulae combined with some well known results are used to calculate $\{m\} \otimes \{\lambda\}$ for all partitions (λ) of n. The results are tabulated for all cases for which $mn \leqq 20$.

D. E. Littlewood (Bangor, Wales).

Nôno, Takayuki. A theorem on the unitary groups over rings. J. Sci. Hiroshima Univ. Ser. A. 17, 173–179 (1953).

Dans un travail récent (apparemment ignoré de l'auteur) le rapporteur a montré comment se généralise le théorème classique sur l'isomorphisme entre le groupe unitaire quaternionique et l'intersection du groupe unitaire (sur les nombres complexes) et du groupe symplectique (sur les nombres complexes) de dimension double; cela vaut pour des groupes unitaires sur des corps non commutatifs beaucoup plus généraux [Acta Math. 87, 175–242 (1952); ces Rev. 14, 239]. L'auteur étudie un cas encore plus général, le corps étant remplacé par un anneau ayant un élément unité; les méthodes et les résultats sont tout à fait analogues. La terminologie de l'auteur est parfois surprenante: il parle, par exemple, d'un espace vectoriel dont les scalaires sont, non dans un anneau, mais dans un autre espace vectoriel! Le nom de "groupe symplectique" qu'il attribue à un groupe unitaire sur un anneau non commutatif est aussi assez fâcheux. *J. Dieudonné* (Evanston, Ill.).

Cotlar, M., and Ricabarra, R. On the existence of characters in topological groups. Amer. J. Math. 76, 375–388 (1954).

The authors find a necessary and sufficient condition for a commutative topological group G to be maximally almost periodic. Specifically, their principal theorem is as follows. An open set is called big if a finite number of its translates cover G. Theorem: If $x \neq e$ then there exists a continuous character χ on G such that $\chi(x) \neq 1$ if and only if there exists a symmetric neighborhood U of the identity e which is big and is such that U^2 does not contain x. The authors also prove generalizations of two theorems concerning the spectrum of a function. In proof of the main theorem, the authors use the finitely-additive invariant measure furnished by the Hahn-Banach theorem to construct from such a neighborhood U a certain continuous positive-definite function f. Then f is placed in a compact convex set C of functions which is invariant under a suitable operation, and the extreme points of C, guaranteed by the Krein-Milman theorem, are shown to be characters. If the functions of C are first convoluted with a suitable function, then C becomes a uniformly equicontinuous family and the extreme characters are now all continuous. *L. H. Loomis.*

Sakai, Shôichirô. On the representations of semi-simple Lie groups. Proc. Japan Acad. 30, 14–18 (1954).

Sakai, Shôichirô. On infinite-dimensional representations of semi-simple Lie algebras and some functionals on the universal enveloping algebras. I. Proc. Japan Acad. 30, 305–312 (1954).

The author states a series of definitions and theorems about the representations of semi-simple Lie groups and Lie algebras, which are very closely related to the work of Harish-Chandra. Some proofs are given or indicated, but the author states that the details will be published elsewhere.

F. I. Mautner (Princeton, N. J.).

Harish-Chandra. The Plancherel formula for complex semisimple Lie groups. Trans. Amer. Math. Soc. 76, 485–528 (1954).

Let G be a separable locally compact unimodular group. Let U denote the regular representation of G. Let $U = \int U^z d\mu(z)$ denote the canonical direct integral decomposition of U into factor representations. The measure μ is only determined up to an equivalence but, if U is of type I, μ becomes unique on demanding further that for a suitable large class of complex-valued functions f on G we have $f(e) = \int \text{Trace } (L_f{}^z) d\mu(z)$, where L^z is the irreducible representation of which U^z is a multiple and L_f is defined by the equation: $(L_f(\phi), \psi) = \int f(x)(L_x(\varphi), \psi)dx$. The existence of a μ with this additional property has been established by Segal and by Mautner.

For many particular groups G the set $\hat{G}$ of equivalence classes of irreducible unitary representations has sufficient structure so that it makes sense to ask: Which measure in $\hat{G}$ is the above μ? The present paper is devoted to answering this question when G is a complex connected semi-simple Lie group. It is written in such a way that a large part of the discussion applies also to the real case, where the problem has so far been solved only in a special case.

The author introduces a family ω of equivalence classes of unitary representations based on two parameters ν and λ, where ν lies in a certain real vector space $\mathfrak{F}_+$ and λ lies in a certain countable subset $\mathfrak{F}_-$ of $\mathfrak{F}_+$. He shows that for almost all ν in $\mathfrak{F}_+$ (with respect to Lebesgue measure) $\omega(\nu, \lambda)$ is irreducible for all λ and constructs a subset $\mathfrak{F}_+{}^0$ of $\mathfrak{F}_+$ such that for $\nu, \lambda \in \mathfrak{F}_+{}^0 \times \mathfrak{F}_-$ the $\omega(\nu, \lambda)$ are distinct and irreducible. Now $\mathfrak{F}_+{}^0 \times \mathfrak{F}_-$ has a natural measure (up to a multiplicative constant), namely the direct product of the restriction to $\mathfrak{F}_+{}^0$ of Lebesgue measure in $\mathfrak{F}_+$ with the counting measure in $\mathfrak{F}_-$. This measure transferred to $\hat{G}$ via the mapping ω is the measure required. The proof, which is rather long, does not depend upon the abstract theorem of Segal and Mautner. A detailed summary has been published elsewhere [Proc. Nat. Acad. Sci. U. S. A. 37, 813–818 (1951); these Rev. 13,

533]. In the review of the summary will be found references to previous work of Gelfand and Neumark on special cases of the author's theorem. *G. W. Mackey.*

***Chern, Shiing-Shen. Pseudo-groupes continus infinis.** Géométrie différentielle. Colloques Internationaux du Centre National de la Recherche Scientifique, Strasbourg, 1953, pp. 119–136. Centre National de la Recherche Scientifique, Paris, 1953.

This lecture concerns the infinite continuous groups introduced by Lie and E. Cartan, i.e. transformation groups defined by the general integrals of some systems of partial differential equations (e.g. all complex-analytic transformations). The starting point is the reduction of the structure group of the tangent bundle of a manifold from the general linear group to some subgroup G, thus defining a G-structure on V, with bundle space B_G. The structure equations lead to first invariants; if they are constant the G-structure is called integrable (for an almost-complex structure these are the integrability conditions [cf. Ehresmann, Proc. Internat. Congress Math., Cambridge, Mass., 1950, v. 2, Amer. Math. Soc., Providence, R. I., 1952, pp. 412–419; these Rev. 13, 574] which make the structure true-complex). The construction of a G-structure can be considered as a special case of solving a differential system. The concept of involutive system leads to a definition of involutive groups (the involutive character of the systems in question depends only on the group G; definitions are too complex to be given here). The orthogonal groups are not involutive. Any involutive integrable structure defines a pseudo-group of transformations; this can be considered as generalization of the converse of Lie's third theorem. Special cases are: (left) translations in Lie groups, complex-analytic transformations, contact transformations. The bundle space B_G of a G-structure has an induced structure with group G_1, called the deduced group of G. It is recalled that Cartan, using these and other concepts, has determined the (complex) simple infinite groups. Some global problems, relating to the question whether a manifold can carry a structure corresponding to one of these groups, are considered; they lead to conditions on the Stiefel-Whitney classes.

It is stated that the topics discussed in this lecture form only a particular case of those considered by E. Cartan, and that various generalizations seem possible. As an isolated fact it is noted that the tangent bundle of a manifold can never be homeomorphic to a sphere (E. H. Spanier).

H. Samelson (Ann Arbor, Mich.).

Horie, Nobuo. On some properties of trajectories of the group-spaces. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 28, 169–178 (1954).

Let $C(t)$ be a one-parameter subgroup of a connected Lie group G; then the author shows $C(t)=aC(t')a^{-1}$ holds for suitable t' and all $a \in G$ if and only if C is normal in G. If C is not normal in G, there may be subsets (the normalizer of C in G) whose elements a have the desired property. This case is considered in some detail. C can be a closed path in G, and have the property $C(t)=aC(t')a^{-1}$ only if $t=t'$.

A. Nijenhuis (Princeton, N. J.).

Mostert, Paul S. On a locally compact group acting on a manifold. Proc. Amer. Math. Soc. 5, 769–770 (1954).

The following theorem is proved: If G is a locally compact group which acts transitively on a compact manifold M, and if G_x is connected, then G contains a compact subgroup which acts transitively on M. When G is a Lie group this was proved by the reviewer [same Proc. 1, 467–469 (1950); these Rev. 12, 242]. *D. Montgomery* (Princeton, N. J.).

Kasuga, Takashi. On the isomorphism of topological groups. Proc. Japan Acad. 29, 435–438 (1953).

The author proves a number of theorems on the identification of topological groups having the same underlying group. Suppose, for example, that the topological groups $G(T_1)$, $G(T_2)$ have topologies T_1, T_2 on the underlying group G. If both are semicompact, and if there exists a topology on the set G which is weaker than both T_1 and T_2, then the two topological groups are identical.

P. A. Smith (New York, N. Y.).

Koch, R. J., and Wallace, A. D. Maximal ideals in compact semigroups. Duke Math. J. 21, 681–685 (1954).

A mob is a (Hausdorff) topological semigroup. Throughout this review, S will denote a compact mob. The main result of this paper is: every proper left (right, two-sided) ideal of S is contained in a maximal proper left (right, two-sided) ideal of S, which is necessarily open. If S is connected, such a maximal ideal is dense in S. If the set E of idempotents of S is contained in a maximal proper (two-sided) ideal J of S, then $S^2 \subset J$. It follows that if $S^2=S$, then $S=SES$. More particularly, if S has a unique idempotent, and $S^2=S$, then S is a topological group.

Next, a short proof is given of a theorem of Wallace [Math. J. Okayama Univ. 3, 1–3 (1953); these Rev. 15, 933], namely, if S has a unit and is an indecomposable continuum, then S is a topological group. Finally, some results are given that relate maximal proper ideals of S and elements x of S such that $xS=S$. In particular, if S has a unit and is not a group, the maximal proper ideal of S is unique and is the complement of the largest subgroup of S containing the unit element.

Reviewer's remarks: 1. A special case of the unpublished result of Wallace quoted on p. 684 was given by him earlier [ibid. 3, 23–28 (1953); these Rev. 15, 933]. 2. Two papers of Š. Schwarz on maximal ideals of (discrete) semigroups [Čehoslovack. Mat. Ž. 3(78), 139–153, 365–383 (1953); these Rev. 15, 850] have features in common with the paper under review. *M. Henriksen* (Lafayette, Ind.).

NUMBER THEORY

Duarte, F. J. Tables of logarithms of the prime factorials from 2 to 10007 to 33 decimals. Estados Unidos de Venezuela. Bol. Acad. Ci. Fís. Mat. Nat. 15, no. 47, 3–54 (1952). (Spanish)

The table gives 33 decimal place values of the sum $\theta_1(x)$ of the common logarithms of the primes not exceeding x for every prime value of $x \leqq 10007$. A small table gives 3 decimal values $\theta(y)/y=(\log_e 10)\theta_1(y)/y$ for

$$y=50, 100(100)1000(1000)10000.$$

For $y=10000$ the value is .990. According to the prime-number theorem $\theta(y)/y \to 1$ as $y \to \infty$. The table is photolithographed from typescript. *D. H. Lehmer.*

Xeroudakes, George, and Moessner, Alfred. A theorem of the elementary arithmetic. Jber. Deutsch. Math. Verein. 57, Abt. 1, 89–92 (1954).

The following result is proved. If $n \geqq 1$ and N has at least one prime divisor of the form $4k+1$, then there exist

integers x, y, z, t such that $2(N^n)^m = x^m + y^m + z^m + t^m$ $(m = 1, 2, 3)$.

L. Carlitz (Durham, N. C.).

Kesava Menon, P. On the equation

$$y^3 = x^3 - 3\lambda\mu x - \lambda^3 a - \mu^3 a^{-1}.$$

Math. Student **22**, 85–88 (1954).

The equation of the title has no known rational solutions x, y for arbitrarily given values of a, λ and μ. From one known solution others can be found, for example by the tangent and chord process. This paper gives a procedure by means of algebraic transformations for finding other solutions from a given one. As corollaries, parametric solutions are obtained for two special equations where a particular solution is obvious. *I. Niven* (Eugene, Ore.).

Nicol, C. A. A note concerning the quotient $(r^{p-1}-1)/p$. Amer. Math. Monthly **61**, 562–563 (1954).

Write $q(r)$ for $(r^{p-1}-1)/p$, where p and r are primes, p odd. It is proved that if r is a primitive root of p, then $q(r) = (r-1)\{e(r) + \frac{1}{2}(p-1)\}$, where $e(r)$ denotes the exponent of the highest power of r which divides $q(r)!$. This result is used to relate the sum of the primitive roots of p to the sum of their $(p-1)$th powers. A formula for $q(r)$ is given also in case the prime r is not a primitive root, in terms of d, the exponent to which r belongs modulo p.

I. Niven (Eugene, Ore.).

Carlitz, L. Congruences for the solutions of certain difference equations of the second order. Duke Math. J. **21**, 669–679 (1954).

The author discusses the difference equation

$$u_{n+1} = nu_n + cu_{n-1},$$

where u_0, u_1 and c are assigned integers. He first defines $\Delta^r u_n$ by means of

$$(1) \qquad \Delta^r u_n = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} c^{m(r-s)} u_{n+2sm},$$

and proves that $\Delta^{2r-1} u_n \equiv \Delta^{2r} u_n \equiv 0 \pmod{m^r}$ for $r \geqq 1$, $(c, m) = 1$, m odd and all n; for even m the modulus is replaced by $2^{-r+1} m^r$. He next considers the equation

$$(2) \qquad u_{n+1} = nu_n + bcu_{n-1} + b^n + c^n \quad (bc \neq 0),$$

where $u_0 = 0$, $u_1 = 1$ and b, c are indeterminates or rational numbers such that b, c, b^{-1}, c^{-1} are all integral (mod m); also he assumes that $b \equiv c \pmod{p}$ for every prime p dividing m. Then he shows that $u_{n+m} \equiv b^m u_n \equiv c^m u_n \pmod{m'}$, where m' is defined by $m = \prod_{p|m} p^s$, $m' = \prod_{p|m} p^{s-1}$. Defining $\Delta^r u_n$ as in (1) but with u_{n+2sm} replaced by u_{n+sm}, he shows that $\Delta^{2r} u_n \equiv \Delta^{2r+1} u_n \equiv 0 \pmod{m^r m'}$ for $r \geqq 1$, all n and odd m; for even m the modulus is replaced by $2^{-r+1} m^r m'$. Finally the author considers briefly an equation somewhat more general than (2). *A. L. Whiteman* (Los Angeles, Calif.).

Carlitz, L. Congruence properties of the ménage polynomials. Scripta Math. **20**, 51–57 (1954).

$$U_n(t) = \sum_{k=0}^{n} \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!(t-1)^k$$

étant la fonction génératrice des nombres $u(n, k)$ rattachés au problème des ménages de Lucas, généralisé, l'auteur avait montré dans un précédent travail [Duke Math. J. **19**, 549–552 (1952); ces Rev. **14**, 346] que la congruence $U_{m^2+n} \equiv (t-1)^{m^2} U_n$ est vérifiée pour toute valeur du module m. Ici, il est prouvé (1) que la congruence $U_{m+n} \equiv (t-1)^m U_n$ est vraie pour un certain module m'; (2) que, pour: $p^j \leqq k < p^{j+1}$ et pour le module (p^{s-1}, p^{s-j}) on a:

$$u(p^s+n, k) \equiv -u(n, k); \quad u(p^s+n, p^s+k) \equiv u(n, k);$$

(3) que

$$\Delta^{2r} U_n \equiv \Delta^{2r+1} U_n \equiv 0 \pmod{m^r m'} \quad (r \geqq 1),$$

ou (mod $2^{1-r} m^r m'$) si m est pair, avec

$$\Delta f(n) = f(n+m) - (t-1)^m f(n); \quad \Delta^r f(n) = \Delta \cdot \Delta^{r-1} f(n).$$

Les valeurs négatives de n sont introduites et les résultats sont étendus au cas où les ménages sont assis à une table non circulaire. *A. Sade* (Marseille).

Lehmer, D. H. A sieve problem on "pseudo-squares." Math. Tables and Other Aids to Computation **8**, 241–242 (1954).

Lorentz, G. G. On a problem of additive number theory. Proc. Amer. Math. Soc. **5**, 838–841 (1954).

If A is a set of natural numbers, then $A(n)$ denotes the number of $a \in A$ such that $a \leqq n$. Two sets A, B of natural numbers are said to be complementary if $A+B$ contains all sufficiently large numbers. The main result proved in the present paper is as follows: for each infinite set A there exists a complementary set B such that

$$(*) \qquad B(n) \leqq C \sum_{k=1}^{n} \frac{\log A(k)}{A(k)};$$

here C is an absolute constant, and the terms of the sum with $A(k) = 0$ are to be replaced by 1. From this it follows almost at once that each infinite set A has a complementary set of asymptotic density zero, a result previously conjectured by P. Erdös. A further theorem that can be deduced from the proof of (*) is as follows: if $a_1, \cdots, a_l$ is a set of incongruent residues (mod n), there is another set of residues $b_1, \cdots, b_k$, with $k \leqq Cn \log l/l$, such that each residue (mod n) is of the form $a_i + b_j$. *L. Mirsky* (Sheffield).

Duparc, H. J. A. Periodicity properties of recurring sequences. I. Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indagationes Math. **16**, 331–342 (1954).

This paper is the first of two papers which constitute a redevelopment of the author's dissertation [Univ. of Amsterdam, 1953; these Rev. **15**, 200]. It contains the general algebraic properties necessary for the future consideration of the recurring series mentioned in the title. The 29 theorems of the paper are mostly concerned with generalizations of Fermat's theorem and the periodicity of powers with respect to a modulus. The moduli considered are elements of a commutative ring with unique factorisation in which the residue classes form a finite field and the extension of such systems to polynomial fields.

D. H. Lehmer (Berkeley, Calif.).

Steuerwald, Rudolf. Ein Satz über natürliche Zahlen N mit $\sigma(N) = 3N$. Arch. Math. **5**, 449–451 (1954).

The following theorem is proved. If $6 | N$ and $\sigma(N) = 3N$, then N contains one of the primes 2, 3 to the first power and the other to a higher power. It is stated that by means of this theorem it can be shown that $2^3 \cdot 3 \cdot 5$, $2^5 \cdot 3 \cdot 7$, $2^3 \cdot 3 \cdot 11 \cdot 31$ are the only numbers with less than 5 prime factors that satisfy $\sigma(N) = 3N$. *L. Carlitz* (Durham, N. C.).

Bellman, Richard, and Shapiro, Harold N. The distribution of squarefree integers in small intervals. Duke Math. J. 21, 629–637 (1954).

Let $Q(u, v)$ denote the number of square-free integers q such that $u<q\leqq v$. It is proved first that if $\phi(x)$ is any function such that $\phi(x)\to\infty$ as $x\to\infty$, then for almost all n the interval $(n, n+\phi(n))$ contains a square-free integer. While proved independently, this is a consequence of the following theorem. If $\phi(n)$ is any strictly monotone function such that $\phi(x)\to\infty$ as $x\to\infty$, then $Q(n, n+\phi(n))$ has the normal order $6\phi(n)/\pi^2$. Finally this result is sharpened as follows. If $k(n)\to\infty$ as $n\to\infty$, then for almost all n

$$\frac{6}{\pi^2}\phi(n)-k(n)\phi^{1/2}(n)\leqq Q(n, n+\phi(n))\leqq\frac{6}{\pi^2}\phi(n)+k(n)\phi^{1/2}(n),$$

provided $\phi(n)$ satisfies certain additional requirements.

L. Carlitz (Durham, N. C.).

Rieger, G. J. Zu Linniks Lösung des Waringschen Problems: Abschätzung von $g(n)$. Math. Z. 60, 213–234 (1954).

The author follows Linnik's elementary solution of Waring's Problem, as presented in Chapter 3 of Khintchine's "Three pearls of number theory" [2nd ed., OGIZ, Moscow-Leningrad, 1948; English translation published by Graylock Press, Rochester, N. Y., 1952; these Rev. 11, 83; 13, 724]. He obtains an explicit upper bound for $g(n, f, \delta)$, which is defined as the least g such that every positive integer is representable as a sum of at most g numbers, each of which is either 1 or a value of $f(x)$, where $f(x)$ is a given polynomial of degree n with integral coefficients and x runs through any sequence of positive integers of density δ (in the sense of Schnirelmann). For the classical case when x runs through all positive integers, the upper bound obtained is naturally very poor compared with known results.

H. Davenport (London).

Beyer, Gudrun. Über eine Klasseneinteilung aller kubischen Restcharaktere. Abh. Math. Sem. Univ. Hamburg 19, no. 1–2, 115–116 (1954).

In a recent note [Math. Tables and Other Aids to Computation 7, 133–134 (1953); these Rev. 14, 1126] von Neumann and Goldstine reported the results of extensive numerical investigation into the so-called Kummer conjecture. Kummer classified all primes $p=6n+1$ into three classes and conjectured that their densities are 1/2, 1/3 and 1/6 from observations on the first 45 primes. The work of von Neumann and Goldstine extended to the 611 primes p under 10000 and gave densities .4452, .3290, .2258 which do not differ much from 4/9, 3/9 and 2/9. The author points out that the problem may be generalized to the case in which p is no longer a prime by multiplying the appropriate Gauss sums together. By considering 1115 cases, both prime and composite, under 7057 the author comes up with densities .4458, .3318, .2224. Unfortunately the author does not specify which composite cases were considered so that the reader is a little uncertain of the new rules of the game. The author has investigated similarly 619 cases less than 1000 of the biquadratic Gauss sums but he feels that he has insufficient evidence upon which to base a conjecture. Hence no results are given.

D. H. Lehmer (Berkeley, Calif.).

Prachar, K. Über ein Resultat von A. Walfisz. Monatsh. Math. 58, 114–116 (1954).

The author shows that almost all primes are "isolated" from their neighbouring primes. More precisely, let $f(x)$ be a function monotonic for large x and such that $f(x)\to\infty$, $\log x/f(x)\to\infty$ as $x\to\infty$. Then the number $s(x)$ of primes $p\leqq x$, for which every q with $0<|q-p|\leqq\log p/f(p)$ is composite, satisfies the relation $s(x)\sim x/\log x$. This result sharpens an earlier result of the same type due to A. Walfisz [Doklady Akad. Nauk SSSR (N.S.) 90, 711–713 (1953); these Rev. 15, 102]. The main tool used in the proof is the Brun-Schnirelmann upper estimate of the number $A(x, y)$ of primes $p\leqq x$ for which $p+y$ is also a prime.

L. Mirsky (Sheffield).

Forman, William, and Shapiro, Harold N. Abstract prime number theorems. Comm. Pure Appl. Math. 7, 587–619 (1954).

Let G be a free Abelian group on a countable number of generators g_i ($i=1, 2, \cdots$), and N a homomorphism of G into the positive rationals such that the g_i (and so all integral words w of G) have integral images Ng_i (and Nw). Let G' be a subgroup of finite index $h=[G: G']$, and suppose that, for all cosets H of G/G',

$$B_H(x)\equiv\sum_{\substack{Nw\leqq x\\ w\in H}}1=c_Hx+O(x^\theta),\quad 0\leqq\theta<1,\ c_H\geqq 0;\quad c=\sum_H c_H\neq 0.$$

Then, for each H,

$$\pi_H(x)\equiv\sum_{\substack{Ng\leqq x\\ g\in H}}1=\frac{d_Hx}{\log x}+o\left(\frac{x}{\log x}\right).$$

This abstract theorem includes the classical prime number theorems for arithmetical progressions and ideal classes. Thus, to cover the case of primes in arithmetical progressions of difference A, where A is a given positive integer, we identify the g_i with the (positive) primes not dividing A [taking Ng_i equal to g_i, presumably] and take G' to be the set of positive rational numbers $r=m/n$ of G for which $r\equiv 1 \pmod A$ [i.e. $(m, A)=(n, A)=1$, $m\equiv n \pmod A$]. In this case the c_H are all equal, and when this is so we have $d_H=1/h$ for each H (so that the g_i "equi-distribute" over the cosets H). But this "classical" situation is very special. The authors give an exhaustive analysis of the general case (i.e. a complete specification of the values of the d_H). This hinges on a certain between-group L ($G'\subset L\subset G$), but the possibilities are too numerous for a short summary. When $\theta<\frac{1}{2}$, however, some of these possibilities are excluded by an argument modelled on one of the function-theoretical proofs of the non-vanishing of the Dirichlet L-functions $L(s, \chi)$ at $s=1$; and the situation then is that $d_H=1/[L: G']$ if $H\in L/G'$ (i.e. $H\subset L$) and $d_H=0$ otherwise (so that the g_i are "almost all" in L and "equi-distribute" among the cosets of G' in L).

The proof of the general abstract theorem follows the lines of Selberg's elementary proof of the prime number theorem, as modified by Shapiro. The various possibilities are illustrated by examples based on ordinary integers and primes. In the examples designed to show that the simplification when $\theta<\frac{1}{2}$ is not universal, the Riemann hypothesis is assumed [though it would be sufficient to assume $\Theta<1$, where Θ is the upper bound of the real parts of the zeros of Riemann's $\zeta(s)$].

[Remarks by the reviewer: (1) The definition of "Dirichlet density" on p. 600 is subject to an "existence" qualification, but this is not referred to in later developments. Thus, in the proof of Lemma A on p. 601, some a priori reason should be given for the existence of all the densities used, or failing this the argument should be conducted

initially in terms of "upper and lower densities" (suitably defined). (2) The proof of Theorem I on pp. 614–616 (i.e. the simplification when $\theta<\frac{1}{2}$) is obscure. The statement at the bottom of p. 615 that $g(s)$ is regular in the half-plane $\sigma>\theta$ may be vitiated by zeros of $\zeta_Q(2s)$; but Landau's theorem only calls for regularity along the portion $s\geqq\frac{1}{2}$ of the real axis, and this can be asserted with safety (on the assumption that $L(1,\chi)=0$). The statement near the top of p. 616 that $h(s)$ is regular and $h(s)\neq 0$ for $\sigma\geqq\frac{1}{2}$ seems to be based on the fact that $h(s)=\prod_g\{1-\eta_g(s)\}$, where $\eta_g(s)$ is regular, $\eta_g(s)\neq 1$, and $|\eta_g(s)|<K(Ng)^{-2\sigma}$, in $\sigma\geqq\frac{1}{2}$; but this by itself only permits a conclusion in $\sigma>\frac{1}{2}$ (and incidentally it is not clear why the list of exceptional g given in the bracket should not include those for which $Ng=3$). However, if the definition of $g(s)$ is modified so that only one factor $\zeta_Q(2s)$ occurs in the denominator in (6.7), the classical situation seems to be paralleled, so that Landau's theorem can be applied directly to $g(s)$ (with no exceptional g's).] *A. E. Ingham* (Cambridge, England).

Ward, Morgan. Cyclotomy and the converse of Fermat's theorem. Amer. Math. Monthly **61**, 564 (1954).

A simple proof is given of the following theorem stated without proof by A. Hurwitz [Math. Werke, Bd 2, Birkhauser, Basel, 1933, p. 747]: Let $Q_m(x)$ be the polynomial whose roots are the distinct primitive mth roots of unity. If, for some integer a, $Q_{n-1}(a)$ is divisible by n, then n is a prime. This condition on n is shown to be equivalent to Lucas' converse of Fermat's theorem: If n divides $a^{n-1}-1$ but not a^d-1 where d is a proper divisor of $n-1$, then n is a prime. *D. H. Lehmer* (Berkeley, Calif.).

Lehmer, Emma. On cyclotomic numbers of order sixteen. Canadian J. Math. **6**, 449–454 (1954).

If $(i,j)_8$ is the number of solutions of

$$g^{8s+i}+1\equiv g^{8s+j} \pmod p,$$

then Dickson [Amer. J. Math. **57**, 391–424 (1935)] showed that $64(i,j)_8$ is expressible for each i, j as a linear combination with integral coefficients of p, x, y, a, b where $p=x^2+4y^2=a^2+2b^2=8n+1$, $a\equiv b\equiv 1 \pmod 4$. The question has been raised whether constants $\alpha,\beta,\gamma,\delta,\epsilon$ can be found such that

$$(*)\qquad 256(i,j)_{16}=p+\alpha x+\beta y+\gamma a+\delta b+\epsilon$$

at least for some i, j. The present paper describes the results of an experiment undertaken on the National Bureau of Standards Automatic Computer. Eight primes p of the form $32n+1$ for which 2 is not a quartic residue were selected. It was found that (*) is not satisfied for any $(i,j)_{16}$. A similar calculation subsequently undertaken for primes of the form $32n+17$ also led to a negative result. *L. Carlitz* (Durham, N. C.).

Nicol, C. A., and Vandiver, H. S. A von Sterneck arithmetical function and restricted partitions with respect to a modulus. Proc. Nat. Acad. Sci. U. S. A. **40**, 825–835 (1954).

The function referred to in the title is the sum of the kth powers of the primitive nth roots of unity, otherwise known as Ramanujan's sum. This sum depends only on n and D, the greatest common divisor of n and k. An explicit formula for this sum was given by Hölder [Prace Mat.-Fiz. **43**, 13–23 (1936)]: $\Phi(k,n)=\mu(n/D)\phi(n)/\phi(n/D)$, where ϕ and μ are the functions of Euler and Möbius. This function had occurred in work by von Sterneck [Akad. Wiss. Wien, S.-B. IIa. **111**, 1567–1601 (1902)] on partitions modulo n. The present paper gives a number of properties of $\Phi(k,n)$ based on its connection with roots of unity and using the divisor calculus. For example two inversion formulae are given:

$$g_n(s)=\sum_{d\mid n}f(d)\Phi(s,n/d) \quad \text{implies} \quad nf(n)=\sum_{s=1}^{n}g_n(s)$$

and

$$G_n(d)=\sum_{k=1}^{n}F(k)\Phi(k,d) \quad \text{implies} \quad nF(n)=\sum_{d\mid n}g_n(d).$$

The sum $\sum_{s=1}^{n}[n/s]\Phi(k,s)$ is the sum of those divisors of k which do not exceed n, while the sum

$$n^{-1}\sum_{d\mid n}\{\Phi(d,n)\}^m\Phi(s,n/d)$$

gives the number of solutions of the congruence

$$x_1+x_2+\cdots+x_m\equiv s \pmod n$$

in which the x's are taken prime to n. Some of these results had been given without proof by Nicol [Proc. Nat. Acad. Sci. U. S. A. **39**, 963–968 (1953); these Rev. **15**, 105]. The function $\Phi(k,n)$ has been discussed recently by Anderson and Apostol [Duke Math. J. **20**, 211–216 (1953); these Rev. **14**, 951]. *D. H. Lehmer* (Berkeley, Calif.).

Vandiver, H. S. On trinomial equations in a finite field. Proc. Nat. Acad. Sci. U. S. A. **40**, 1008–1010 (1954).

Let g denote a multiplicative generator of the non-zero numbers of $GF(p^n)$, and let $(i,j)_m$ denote the number of solutions s, t of $g^{ic+s}+g^{jc+t}=1$, where $p^n-1=cm$, $0\leqq i<c$, $0\leqq j<m$, $0\leqq s<m$, $0\leqq t<c$. The following two theorems are proved. I. If $b\not\equiv 0 \pmod c$, $d\not\equiv 0 \pmod c$, a arbitrary, $c=c_1m$, then

$$\sum_{i=0}^{m-1}(b,i)_m(b-a,i-d)_m=\sum_{j=0}^{c-1}\sum_{d}(d,j)_c(d-a,j-b)_c,$$

where d ranges over the values $d_1, d_1+m, \cdots, d_1+(c_1-1)m$. This relation also holds for $b\not\equiv 0 \pmod c$, $d\equiv 0 \pmod c$, $a\not\equiv 0 \pmod c$. II. If $(b,h)_m=m$, $(b,i)_m=0$ for $i\not\equiv h \pmod m$, $b\not\equiv 0 \pmod c$, then $(d,j)_c(d,j-b)_c=0$, and

$$\sum_{j=0}^{c-1}\sum_{d=1}^{c_1-1}(dm,j)_c(dm,j-b)_c=m^2-m.$$

L. Carlitz (Durham, N. C.).

Carlitz, L. The number of solutions of some equations in a finite field. Portugaliae Math. **13**, 25–31 (1954).

The title of this paper refers principally to the equation $(x+y+z-b)^2=2axyz$ taken over a field k with q elements, q being odd, and to the corresponding equation in four variables. It is shown that the number of solutions (x,y,z) of the first equation subject to the condition $a\neq 0$ is furnished by $q^2+1+q\psi(ab(ab-2))$, where $\psi(a)=+1$, -1 or 0 according as a is a square, a non-square or zero in k. The number of solutions of the second equation is exhibited in terms of sums of the Jacobsthal type. *A. L. Whiteman.*

Carlitz, L. Invariant theory of systems of equations in a finite field. J. Analyse Math. **3**, 382–413 (1954).

Let $GF(q)$ denote a fixed finite field and let k, r be fixed integers $\geqq 1$. Consider the set T of transformations ϕ: $\xi_i=\phi_i(\eta_1,\cdots,\eta_r)$ $(i=1,\cdots,r)$ possessing an inverse; the numbers ξ_i, $\eta_i \in GF(q)$ and the ϕ_i are polynomials with coefficients in $GF(q)$. The set of such transformations form a group isomorphic with the symmetric group on q^r letters.

Two polynomials f, g with coefficients in $GF(q)$ are defined as equal if $f(\xi_1, \cdots, \xi_r) = g(\xi_1, \cdots, \xi_r)$ for all $\xi_i \in GF(q)$. Two sets of polynomials $f_1, \cdots, f_k; g_1, \cdots, g_k$ are equivalent if there exists a transformation ϕ such that $\phi f_i = g_i$ for $i = 1, \cdots, k$. By means of this equivalence relation sets of polynomials are separated into a certain number of classes. The chief purpose of this paper is to discuss properties of the system $f_i(\xi_1, \cdots, \xi_r) = \alpha_i$ $(i = 1, \cdots, k)$ that are invariant under the group of transformations T. Let $N[f, \alpha]$ denote the number of solutions $\xi_1, \cdots, \xi_r \in GF(q)$ of this system. It is shown that the numbers $N[f, \alpha]$ form a complete set of invariants in the following sense: Two polynomial sets $f_1, \cdots, f_k; g_1, \cdots, g_k$ are equivalent if and only if $N[f, \alpha] = N[g, \alpha]$ for all $\alpha_1, \cdots, \alpha_k$. Moreover, any invariant of the set $f_1, \cdots, f_k$ (with rational values, say) can be exhibited as a polynomial in the numbers $N[f, \alpha]$ with rational coefficients. A second set of fundamental invariants is also constructed. The case $k = 1$ has been treated in an earlier paper with a similar title [Trans. Amer. Math. Soc. 75, 405–427 (1953); these Rev. 15, 291]. The present paper, however, is self-contained and develops with considerable detail several new related topics and applications.

A. L. Whiteman (Los Angeles, Calif.).

Remak, Robert. Über algebraische Zahlkörper mit schwachem Einheitsdefekt. Compositio Math. 12, 35–80 (1954).

A finite algebraic number field K is called a field with units defect if a proper subfield K' of K contains the same number of independent units as K does. More precisely, such a field K is said to have strong or weak units defect according as every unit of K is contained in K' or not. In a previous paper [Compositio Math. 10, 245–285 (1952); these Rev. 14, 952] the author proved the following formula for a field K without units defect:

$$\log|D| \leqq n \log n + \frac{R}{R^*} \log 2 \frac{2n(n-1)}{k+1} g^{(k-1)/2}.$$

Here D denotes the discriminant of K, n the degree of K, k the number of independent units in K, and R^* and g are both constants depending solely upon n and k.

In the present paper, the author proves the same formula for a field K with weak units defect by reducing it to the previous case. An essential lemma is the following: Let K be a (totally imaginary) field with units defect and K' a (totally real) subfield of K containing the same number of independent units as K does. Let E and E' be the groups of units of K and K' respectively and let W be the group of roots of unity in K. Then the index $Q = [E : WE']$ is either 1 or 2. The author also studies properties of such fields K, K' and shows in particular that, for a given totally real field K', there exist only a finite number of fields with weak units defect containing K'.

Discussions on the index Q similar to those given in the present paper are also found in H. Hasse, Über die Klassenzahl abelscher Zahlkörper [Akademie-Verlag, Berlin, 1952; these Rev. 14, 141]. *K. Iwasawa* (Cambridge, Mass.).

Fröhlich, A. On fields of class two. Proc. London Math. Soc. (3) 4, 235–256 (1954).

In the present paper the fields of at most class two over the rational field are studied, the class of a field being defined as the length of the central series of the Galois group. As a group of at most class two is the direct product of prime power groups of at most class two it is sufficient to study fields whose degree is a prime power. Only such fields are considered here. A field Λ of class two over the rational number field is then a relative Abelian field over an Abelian field K and can be considered as a class field over K. Relative Abelian fields have been studied, among others, by Hasse, who also considers the special case when the ground field is Abelian. The work of Hasse seems to be unknown to the author. Owing to the special properties of fields of class two of prime-power order, the author can give a more detailed determination of the set of such fields, which are described as relative class fields. When $\mathfrak{A}$ is a module in an Abelian field K, which is invariant under the Galois group of K, the union of all Abelian class fields modulo $\mathfrak{A}$ is called maximal modulo $\mathfrak{A}$. The characterization of fields Λ over K of class two are given in the case when K and Λ are maximal, but any field of class two is a subfield of such a field Λ and can be determined when this is known. The author gives all ideal groups in K (maximal) which belong to fields Λ (maximal over K) which are absolute fields of class two. Then he determines the Galois group of Λ belonging to a given ideal group, which can be made by the help of so called inertia generators and cross coefficients. The characterization of the subfields of Λ is rather complicated.

H. Bergström (Göteborg).

Iseki, Kanesiroo. On the divisor-problem generated by $\zeta^\alpha(s)$. Nat. Sci. Rep. Ochanomizu Univ. 4, 175 (1954).

For real $\alpha > 0$, put

$$\zeta^\alpha(s) = \sum_{n=1}^{\infty} d_\alpha(n) n^{-s} \quad (\mathrm{Re}\, s > 1), \qquad D_\alpha(x) = \sum_{n \leq x} d_\alpha(n).$$

Then

$$(*) \qquad D_\alpha(x) \sim \frac{1}{\Gamma(\alpha)} x (\log x)^{\alpha-1} \quad (x \to \infty).$$

It is stated that (*) is deduced from a lemma which may be proved by the Ikehara-Wiener-Landau method for Tauberian theorems. Details of the proof are not given.

L. Carlitz (Durham, N. C.).

Eichler, Martin. Quaternäre quadratische Formen und die Riemannsche Vermutung für die Kongruenzzetafunktion. Arch. Math. 5, 355–366 (1954).

Let $F(x)$ be a quaternary quadratic form with rational integral coefficients whose g.c.d. $= 1$ and of determinant D. The theta-function

$$\vartheta_F(\tau) = \sum_{-\infty}^{\infty} e^{2\pi i F(x)} = \sum_{n=0}^{\infty} a_F(n) e^{2\pi i n \tau}$$

is a modular function of degree -2 belonging to the subgroup $G(Q)$ of the modular group defined by $ad - bc = 1$, $c \equiv 0 \pmod{Q}$, where Q contains just those primes that divide D. Thus

$$\vartheta_F\left(\frac{a\tau+b}{c\tau+d}\right) = \left(\frac{D}{a}\right)(c\tau+d)^2 \vartheta_F(\tau).$$

Corresponding to the condition $(D/a) = 1$ we have also the subgroup $G_1(Q)$. Put $\vartheta_F(\tau) = E_F(\tau) + S_F(\tau)$, where

$$E_F(\tau) = \sum_{n=0}^{\infty} e_F(n) e^{2\pi i n \tau}, \quad S_F(\tau) = \sum_{n=0}^{\infty} \sigma_F(n) e^{2\pi i n \tau},$$

an Eisenstein series and a cusp form. Thus

$$a_F(n) = e_F(n) + \sigma_F(n).$$

The principal term $\epsilon_P(n)$ is known from the work of Siegel; if $(n, D)=1$ then

$$\epsilon_P(n)=C_P\cdot n\sum_{t|n}\left(\frac{D}{t}\right)t^{-1},$$

where C_P is independent of n. In the present paper it is proved that

$$(*)\qquad |\sigma_P(n)|<\gamma_P(\epsilon)n^{1+\epsilon}\quad(\epsilon>0),$$

where $\gamma_P(\epsilon)$ is independent of n, and indeed for p prime

$$(**)\qquad |\sigma_P(p)|<\gamma_P p^{\frac{1}{2}}.$$

To prove (*) and (**) the author considers a certain field K of modular functions belonging to $G_1(Q)$, the field of constants is the rational field; $\bar{K}$ is the field obtained from K by replacing the field of constants by its algebraic closure. Let K_1 consist of those functions in K that are invariant under $G(Q)$ so that K is of degree 2 relative to K_1; let η denote the generator of the Galois group of K/K_1. Reduction of K (mod p) gives the field $K(p)$ with prime field of characteristic p (for all but a finite number of primes p). The Hecke operators T_n [Math. Ann. 114, 1–28, 316–351 (1937)] define multipliers of $\bar{K}$ which generate a subring M_0 of the ring M of all multipliers of $\bar{K}$. To M_0 corresponds the subset $M_0(p)$ of $M(p)$, the multiplier ring of $\overline{K(p)}$. Denote the image of η, τ_n in $M_0(p)$ by $\eta(p)$, $\tau_n(p)$, respectively; let $\pi(p)$ denote the multiplier of $\overline{K(p)}$ obtained by taking pth powers, and let * denote the Rosati antiautomorphism. Then it is proved that

$$\tau_p(p)=\eta(p)^{\frac{1}{2}(1-(D/p))}\pi(p)+\pi(p)^*;$$

in particular, for D a square

$$\tau_p(p)=\pi(p)+\pi(p)^*.$$

The Riemann hypothesis for the zeta function of $K(p)$ may be stated in the form $|\pi_i(p)|=p^{\frac{1}{2}}$, where the $\pi_i(p)$ are the eigenvalues of the operator $\pi(p)$. This result together with $M_0\cong M_0(p)$ implies (*) and (**).

Now let D be a square. Then the Dirichlet series corresponding to the cusp forms belonging to the group $G(Q)$ have an Euler factorization $Z_p(s)=(1_g-T_pp^{-s}+1_gp^{1-2s})^{-1}$. This implies $|Z_p(s)|^{-1}=N((1-\pi(p)p^{-s})(1-\pi(p)^*p^{-s}))$. On the other hand,

$$\zeta_{K(p)}(s)=\frac{L(p^{-s})}{(1-p^{-s})(1-p^{1-s})}=\frac{N((1-\pi(p)p^{-s})(1-\pi(p)^*p^{-s}))}{(1-p^{-s})(1-p^{1-s})},$$

so that

$$(***)\qquad \prod_p\frac{1}{\zeta_{K(p)}(s)}\sim\frac{1}{\zeta(s)\zeta(s-1)}\prod_p|1_g-T_pp^{-s}+1_gp^{1-2s}|^{-1},$$

where the symbol $\sim$ indicates that a finite number of factors is possibly omitted. The left member of (***) is a meromorphic function and satisfies a functional equation of the usual sort. Thus a conjecture of Hasse is proved for the case of the field K. Previous results in this direction were obtained by Weil [Trans. Amer. Math. Soc. 73, 487–495 (1952); these Rev. 14, 452] for the case $K=k(x, y)$, $ax^m+by^n=1$, and Deuring [Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. 1953, 85–94; these Rev. 15, 779] for the case of an elliptic function field with singular moduli. *L. Carlitz* (Durham, N. C.).

Lenz, Hanfried. Zur Quadratsummendarstellung in relativquadratischen Zahlkörpern. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1953, 283–288 (1954).

The following theorem is proved. Let G denote an algebraic number field, K a quadratic extension of K, and γ any totally positive number of K. Then γ can be given in the form $\gamma=\alpha^2+a_1^2+\cdots+a_m^2$, where α belongs to K and $a_1, \cdots, a_m$ belong to K. The proof of the theorem is elementary. Using Siegel's theorem that every totally positive number of G can be given as a sum of four squares of numbers of G, however, the author can prove that $m=4$.

H. Bergström (Göteborg).

Briggs, W. E. An elementary proof of a theorem about the representation of primes by quadratic forms. Canadian J. Math. 6, 353–363 (1954).

H. Weber [Math. Ann. 20, 301–329 (1882)] proved that every binary quadratic form $ax^2+2bxy+cy^2$ which is properly primitive (i.e., $(a, 2b, c)=1$, $a>0$) is capable of representing infinitely many prime numbers. The author gives an elementary proof of this theorem using the methods of A. Selberg [Ann. of Math. (2) 50, 297–304 (1949); these Rev. 10, 595]. *H. S. A. Potter* (Aberdeen).

Hlawka, Edmund. Grundbegriffe der Geometrie der Zahlen. Jber. Deutsch. Math. Verein. 57, Abt. 1, 37–55 (1954).

This is an excellent exposition of the concepts, methods, and results of the modern geometry of numbers. A rather complete list of references is given, and some of the open questions in the subject are noted. *W. J. LeVeque.*

Pötzl, Hans. Über Sternkörper im dreidimensionalen Raum. Monatsh. Math. 58, 91–102 (1954).

The author generalizes work of Mahler on two-dimensional regions [Proc. London Math. Soc. (2) 49, 128–157 (1946); these Rev. 8, 12] to show that lattice constants of bounded star bodies in three-dimensional space whose surface is composed of a finite number of analytic pieces may be found in a finite number of steps. He is apparently unaware of Mahler's later work [Proc. Roy. Soc. London. Ser. A. 187, 151–187 (1946); Nederl. Akad. Wetensch., Proc. 49, 331–343, 444–454, 524–532, 622–631 (1946); these Rev. 8, 195, 12] in n-dimensions. The author also analyses a method of the reviewer [Quart. J. Math., Oxford Ser. 18, 236–243 (1947); these Rev. 9, 334] for finding the lattice constants of certain two-dimensional regions and shows why it is unreasonable to expect it to generalize to three dimensions. *J. W. S. Cassels* (Cambridge, England).

Šidlovskiĭ, A. B. On transcendentality and algebraic independence of the values of entire functions of certain classes. Doklady Akad. Nauk SSSR (N.S.) 96, 697–700 (1954). (Russian)

C. L. Siegel has introduced the notion of E-functions, and has proved the following theorem [Transcendental numbers, Princeton, 1949, p. 52; these Rev. 11, 330]: Let the E-functions $E_1, \cdots, E_m$ be solutions of a system of m homogeneous linear differential equations of first order, whose coefficients Q_{kl} are rational functions with algebraic numerical coefficients, and suppose that the power-products $E_1^{k_1}\cdots E_m^{k_m}$ $(k_1+\cdots+k_m\leq N)$ form a normal system, for all positive integers N. If α is any algebraic number different from the zeros and poles of any of the Q_{kl}, then the m numbers $E_1(\alpha), \cdots, E_m(\alpha)$ are algebraically independent. In the

present paper, the notion of an irreducible system of E-functions is introduced: $E_1, \cdots, E_m$ are said to form an irreducible system if they are solutions of a system A of linear homogeneous differential equations of the type just specified, none is identically zero, and the relation $\sum_1^m P_k(z)y_k=0$, where the P_k are arbitrary polynomials in z and the y_k are any solutions of A, is possible only in case $P_k y_k=0$ $(1\leqq k\leqq m)$ identically. It is announced that Siegel's theorem has been generalized by dropping the condition of homogeneity and replacing normality by irreducibility. Various applications are mentioned, showing the algebraic independence of values of certain functions satisfying third and fourth order differential equations, for algebraic argument. No proofs are given. *W. J. LeVeque.*

ANALYSIS

*Korowkin, P. P. **Ungleichungen.** Deutscher Verlag der Wissenschaften, Berlin, 1954. 56 pp.

Translation of the author's Neravenstva [Gostehizdat, Moscow, 1952; these Rev. 14, 24].

Bellman, Richard. Inequalities. Math. Mag. **28**, 21–26 (1954).

Expository paper.

Sobel, Milton. On a generalization of an inequality of Hardy, Littlewood, and Pólya. Proc. Amer. Math. Soc. **5**, 596–602 (1954).

The following rather intricate extension of the simple inequality

$$\sum_{i=1}^{n} a_i b_i \geqq \sum_{i=1}^{n} a_i b_{j_i} \geqq \sum_{i=1}^{n} a_i b_{n-i+1}$$

$(a_1\leqq\cdots\leqq a_n;\ b_1\leqq\cdots\leqq b_n;\ j_1, \cdots, j_n$ a rearrangement of $1, \cdots, n)$ [Hardy, Littlewood, and Pólya, Inequalities, Cambridge, 1934, p. 261] is given:

$$v_{r_1}\geqq v_{r_2} \quad \text{if} \quad P^{r_1}>P^{r_2},$$

where P^r is a partition of the set $b_1\leqq\cdots\leqq b_n$ into k disjoint subsets $B_1, \cdots, B_k$, and $P^{r_1}>P^{r_2}$ if there exist left exchanges (transformations interchanging a $b_p\,\varepsilon\,B_s$ with a $b_q\,\varepsilon\,B_t$ $(p<q, s<t)$, i.e. reversing their non-descending order) which carry P^{r_1} into P^{r_2} (this defines a partial order) and v_r is an arbitrary cross-product $\sum_{i=1}^n a_i b_{j_i}$ associating the elements of the B_l $(l=1, \cdots, k)$ belonging to P^r with the elements of the similar subsets A_l of $a_1\leqq a_2\leqq\cdots\leqq a_n$ having the same number of elements as B_l but with the elements a_i remaining in their non-descending order. Some corollaries are also discussed. *J. Aczél* (Debrecen).

Skovgaard, H. On inequalities of the Turán type. Math. Scand. **2**, 65–73 (1954).

The inequality $\Delta_n(x)=[u_n(x)]^2-u_{n-1}(x)u_{n+1}(x)\geqq 0$ is proved for a number of cases partly known by earlier results in connection with the inequality of Turán $[u_n(x)=P_n(x)]$. The tool of the author is the inequality of Laguerre: $[F^{(n)}(z)]^2-F^{(n-1)}(z)F^{(n+1)}(z)\geqq 0$, z real, $n\geqq 1$, satisfied for an entire function of the form

$$C\exp(-\alpha z^2+\beta z)z^r\prod\left(1-\frac{z}{z_m}\right)\exp(z/z_m),$$

where $\alpha>0$, C, β, z_m are real, and $\sum z_m^{-2}<\infty$. In some instances explicit expressions for $\Delta_n(x)$ are found which yield again $\Delta_n(x)\geqq 0$. We mention for $u_n(x)=J_n(x)$ the following interesting formula:

$$\Delta_n(x)=4u_n^2\sum_{k=1}^{\infty}\frac{j_k^2}{(x^2-j_k^2)^2},$$

where j_k are the positive zeros of $J_n(x)$. *G. Szegö.*

Ważewski, T. Une modification due théorème de l'Hôpital liée au problème du prolongement des intégrales des équations différentielles. Ann. Polon. Math. **1**, 1–12 (1954).

The modified theorem is as follows: Let Δ be an open interval of the real axis, with $x=b$ at one end (b may be $\pm\infty$). Let g be a real function which is strictly monotonic on Δ; let F be a Banach-space-valued function defined on Δ. Suppose $F_+'(x)$ and $g_+'(x)$ exist and $g_+'(x)\neq 0$ in $\Delta-Z$, where Z is an at most denumerable set. Suppose $g(x)\to 0$ when $x\to b$, that $\liminf_{x\to b}|F(x)|=0$, and that $\lim F_+'(x_n)/g_+'(x_n)$ exists, $=k$, for each sequence $\{x_n\}$ in $\Delta-Z$ for which $x_n\to b$ and $F(x_n)\to 0$. Then, as $x\to b$, we have $F(x)\to 0$, $F(x)/g(x)\to k$, and if x is restricted to $\Delta-Z$, $F_+'(x)/g_+'(x)\to k$. This theorem is used to prove that, under certain conditions, a solution ψ of the equation $\psi'(x)=G(x,\psi(x))$ (ψ and G vector-valued) on the open interval (a, b) can be extended to a solution valid also at $x=b$. *A. E. Taylor* (Geneva).

Krylov, V. I. Increasing the accuracy of mechanical quadratures. Formulas of Euler's form. Doklady Akad. Nauk SSSR (N.S.) **96**, 429–432 (1954). (Russian)

The author derives some quadrature formulas expressing $\int_a^b p(x)f(x)dx$ in terms of the values $f(x_k)$ and derivatives of f at a and b. [This is not a new idea: cf. Kronecker, S.-B. Preuss. Akad. Wiss. Berlin **1885**, 841–862; Aljančić, Acad. Serbe Sci. Publ. Inst. Math. **2**, 263–269 (1948); these Rev. **10**, 357.] *R. P. Boas, Jr.* (Evanston, Ill.).

Porcelli, Pasquale. Uniform completeness of sets of reciprocals of linear functions. II. Duke Math. J. **21**, 595–597 (1954).

This is a continuation of a study of sets $K:\{(1+k_p x)^{-1}\}_{p=0}^{\infty}$, where $k_p\neq 0$, $k_p\neq k_q$ for $p\neq q$ and k_p is not in the real interval $(-\infty, -1]$, begun in an earlier paper [same J. **20**, 185–193 (1953); these Rev. **15**, 107]. Three earlier characterizations of sets K which are uniformly complete on $[0, 1]$ are supplemented by a fourth, namely, that every function in $L^2[0, 1]$ shall be a limit in the mean of a sequence of linear combinations of functions of K. In contrast to a property of K, a set S of functions, complex-valued and continuous on $[0, 1]$, is exhibited which is not uniformly complete and for which $\varphi\equiv 0$ is a consequence of $\int_0^1 f\,d\varphi=0$ for all f in S, where φ is real-valued and of bounded variation. *W. T. Scott* (Evanston, Ill.).

Serghiesco, Stephan. Sur des invariants différentiels dans le problème du nombre des racines communes à un système d'équations et du nombre de zéros ou des poles d'une fonction de variable complexe. Rend. Circ. Mat. Palermo (2) **2** (1953), 414–441 (1954).

This paper contains fairly elementary, direct proofs of results announced earlier by the author [C. R. Acad. Sci. Paris **186**, 211–213 (1928); **188**, 479–482 (1929); **225**, 485–487 (1947); these Rev. **9**, 179]. These results are similar to

results derived by E. Picard [J. Math. Pures Appl. (4) 8, 5-24 (1892)] from formulas due to Kronecker. The author assumes $f(x, y)$ and $g(x, y)$, functions of the real variables x and y, to be analytic in a certain region of the (x, y)-plane. In any closed contour C drawn in this region, he shows the exact number N of common roots of the equations $f=0$, $g=0$ to be

$$N=\frac{1}{2\pi}\int_C\frac{AdB-BdA}{A^2+B^2}-\frac{1}{4\pi}\lim_{\epsilon\to 0}\iint_S\left\{\frac{\Delta+\epsilon\delta}{[A^2+B^2+(D+\epsilon)^2]^{3/2}}-\frac{\Delta-\epsilon\delta}{[A^2+B^2+(D-\epsilon)^2]^{3/2}}\right\}dxdy,$$

where S is the region bounded by C, and

$$A=\begin{vmatrix} f & g \\ \frac{\partial f}{\partial x} & \frac{\partial g}{\partial x}\end{vmatrix},\quad B=\begin{vmatrix} f & g \\ \frac{\partial f}{\partial y} & \frac{\partial g}{\partial y}\end{vmatrix},\quad D=\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} & \frac{\partial g}{\partial y}\end{vmatrix},$$

$$\Delta=\begin{vmatrix} A & B & D \\ \frac{\partial A}{\partial x} & \frac{\partial B}{\partial x} & \frac{\partial D}{\partial x} \\ \frac{\partial A}{\partial y} & \frac{\partial B}{\partial y} & \frac{\partial D}{\partial y}\end{vmatrix},\quad \delta=\begin{vmatrix} \frac{\partial A}{\partial x} & \frac{\partial B}{\partial x} \\ \frac{\partial A}{\partial y} & \frac{\partial B}{\partial y}\end{vmatrix}.$$

He then generalizes this result to systems of three equations in three unknowns and indicates that it may be further generalized to n equations in n unknowns by the same methods. *M. Marden* (Milwaukee, Wis.).

Theory of Sets, Theory of Functions of Real Variables

Bagemihl, F., and Sprinkle, H. D. On a proposition of Sierpiński's which is equivalent to the continuum hypothesis. Proc. Amer. Math. Soc. 5, 726-728 (1954).

The authors give a strengthening and a correction of a theorem of Sierpiński's and its proof, respectively [cf. Sierpiński, Hypothèse du continu, Warszawa-Lwów, 1934, in particular p. 12, P_3]. Combined with a result of Braun and Sierpiński [Fund. Math. 19, 1-7 (1932), Th. II], the authors establish this theorem: Let M be a set of power $2^{\aleph_\alpha}$ (α any ordinal); then $2^{\aleph_\alpha}=\aleph_{\alpha+1}$ holds if and only if there exists in M an ω_α-sequence of single-valued functions $f_\tau(\tau<\omega_\alpha)$ such that for each $X\subseteq M$ satisfying $|X|>\aleph_\alpha$ and for every $\tau<\omega_\alpha$, except perhaps for less than $\aleph_\alpha$ of them, one has $f_\tau(X)=M$ and $|f_\tau^{-1}(a)|>\aleph_\alpha$ for each $a\in X$ (the strengthening consists just in the terminal addition concerning inverse functions; the correction, for $\alpha=0$, avoids, e.g., that [cf. the book quoted above, p. 13] $\xi^{\omega}_{2k}=2k$ for $\alpha<\omega_1$, $k<\omega_0$, and consequently $|f_{2k}(M)|=1$, as should be allowable in Sierpiński's wording. *Đ. Kurepa* (Zagreb).

Padmavally, K. Generalisation of the ordertype of rational numbers. Revista Mat. Hisp.-Amer. (4) 14, 50-73 (1954).

The main results can be expressed as follows in the notation of the review of another paper by the same author [same Revista (4) 12, 249-265 (1952); these Rev. 15, 18]: Let α denote a limit ordinal, ω_β a regular initial number, and μ an order type. The order of the lexicographically ordered set of sequences $(x_\xi)_{\xi<\alpha}$, where $0\leqq x_\xi\leqq 1$ and, for some $\alpha_0<\alpha$, which may depend on the sequence, x_ξ is equal to some fixed $y\neq 0, 1$ for every $\xi\geqq\alpha_0$, is independent of y, and will be denoted by ρ_α. Every dense subset of ρ_α has order type ρ_α; ρ_α is dense in $\theta((\alpha))$ and can be embedded in $\theta((\alpha))$ as an isolated subset. Every e_β-set [Hausdorff, Math. Ann. 65, 435-505 (1908), p. 487] that can be embedded in $\theta((\omega_\beta))$ as an isolated subset, has order type ρ_{ω_β}. If neither ω_β nor ω_β^* can be embedded in μ, then μ can be embedded in $2((\omega_\beta))$ as an isolated subset. If μ is dense and $c_{\beta\beta}$ is its least symmetric character, then every order type containing neither ω_β nor ω_β^* can be embedded in μ. *F. Bagemihl.*

Kondô, Motokiti. Les éléments quasi-clairsemés. (L'énumération transfinie. I.) Proc. Japan Acad. 30, 66-69 (1954).

Soient R, L l'ensemble ordonné des nombres rationnels et celui des nombres réels respectivement. Pour un $E\subseteq R$ soit ιE l'ensemble des points ordinalement isolés de E; l'A. définit $\delta E=\delta^0 E(=E)-\iota E$, $\delta^{\alpha+1}E=\delta(\delta^\alpha E)$, $\delta^\lambda E=\bigcap_{\xi<\lambda}\delta^\xi E$ pour chaque ordinal λ de seconde espèce. S'il existe un ordinal β vérifiant $\delta^\beta E=0$, E est dit quasi-clairsemé ordinalement (q.c.). Soit X une portion maximale de E semblable á R; si $a=\inf X$, $b=\sup X$, la portion $[a, b]_E$ est appelée "élément épais" de E; une portion q.c. de E ayant au moins 3 points s'appele élément q.c. de E. En commun, les deux espèces d'éléments sont appelés "éléments primaires" de E. Le noyau νE de E s'obtient de E en lui enlevant tous les éléments primaires et en lui ajoutant les extrémités de ceux-ci; $\nu(\nu E)$ s'appelle le noyau épais de E. Chaque portion de E coextensive avec un élément q.c. de νE s'appelle élément composé de E. En ces termes l'A. énonce quelques propositions dont le Th. 2 dit que si $\nu E\neq 0$, E ne contient aucun couple d'éléments composés consécutifs. Aucune bibliographie n'est citée. *Đ. Kurepa* (Zagreb).

Schuff, Hans Konrad. Über die Summation neutraler Zerschlagungen in beliebigen algebraischen Bereichen. Math. Nachr. 11, 295-301 (1954).

Die Arbeit schliesst sich an zwei andere Arbeiten an [vgl. K. Dörge, Math. Nachr. 4, 282-297 (1951); Dörge und Schuff, ibid. 10, 315-330 (1953); diese Rev. 12, 583; 15, 498]. Insbesondere gibt Verf. allgemeine Bedingungen, damit die "kleinste gemeinsame Oberzerschlagung $\overline{\sum Z_\alpha}$ gegebener Zerschlagungen Z_α" mit der "kleinsten neutralen Oberzerschlagung $\sum Z_\alpha$ der neutralen Zerschlagungen Z_α" zusammenfällt: (1) $\overline{\sum Z_\alpha}=\sum Z_\alpha$. Nun besteht (1) für jede Ordnungszahl α und jede monoton steigende $(\alpha+\omega_\alpha)$-Folge F von neutralen Zerschlagungen eines Bereiches B, sobald $\aleph(B)=\aleph_\alpha$ (dabei bedeutet $\aleph(B)$ die kleinste Kardinalzahl, die die Dimension $|\Gamma|$ jeder operation Γ des Bereiches B übertrifft) (Th. 1). Def: "In B gelte der Teilerkettensatz ν-ter Stufe 1., bzw. 2., Art", wenn jede monoton steigende, bzw. fallende, ω_ν-Folge neutraler Zerschlagungen von B stationär ist. Die kleinsten solchen Zahlen ν nennt Verf. "Kettenzahlen" von B (die eine, ν_1, entspricht den steigenden, die andere, ν_2, den fallenden Folgen; der Fall $\nu_1=\nu_2$ kann stattfinden). Zahlen ν_1, ν_2 werden nun durch andere Ausdrucksweise charakterisiert (Basissätze). Besondere Systeme von neutralen Zerschlagungen werden Türme genannt und untersucht. Ein Analog des Bernsteinschen Äquivalensatzes wird formuliert (Th. 11). *Đ. Kurepa.*

Have

*Natanson, I. P. **Theorie der Funktionen einer reellen Veränderlichen.** Akademie-Verlag, Berlin, 1954. xi+478 pp. DM 26.00.

Translation of the author's Teoriya funkcil' veščestvennoĭ peremennoĭ [Gostehizdat, Moscow, 1950; these Rev. 12, 598].

Iseki, Kiyoshi. A construction of two-valued measure on Boolean algebra. J. Osaka Inst. Sci. Tech. Part I. 2, 43–45 (1950).

The author gives a proof by transfinite induction of the known theorem asserting the existence on a Boolean algebra of a finitely-additive two-valued measure which takes the value 1 on any preassigned element. *L. H. Loomis.*

McKean, Henry P., Jr. Notes on some finitely additive measures. J. London Math. Soc. 29, 440–449 (1954).

If α is an ergodic measure-preserving transformation on the unit interval, write M_α for the class of all those sets E (with characteristic function f) for which the limit $m_\alpha(E)=\lim (1/n)\sum_{i=0}^{n-1} f(\alpha^i x)$ exists and is a constant a.e., and let N_α be the subclass of those sets E for which $m_\alpha(E)=0$. The author proves that the restriction of the finitely additive measure m_α to the class of Lebesgue measurable sets is complete non-atomic. If β is another such transformation, α and β are weakly dependent when at almost every point some power of α is equal to some power of β; if at almost every point α itself is equal to some power of β, then α is strongly dependent on β. The following two theorems are typical of the author's remaining results: (1) $N_\alpha=N_\beta$ if and only if α and β are weakly dependent; (2) $M_\alpha\subset M_\beta$ if and only if α is strongly dependent on β. *P. R. Halmos.*

Urbanik, K. Quelques théorèmes sur les mesures. Fund. Math. 41, 150–162 (1954).

This paper is motivated by the measure-theoretic formulation of the Steinhaus cake problem. The problem is to divide a cake among n people, with possibly different standards of value for the various parts of the cake, so that the value of each person's share is, in his own estimation, at least $1/n$ times the value of the whole cake. In the author's mathematization of the problem it is assumed that the n normalized measures that enter are non-atomic, have the same null-sets, and are not all identical. The main theorem asserts that the cake can be divided so that the value of each person's share is, in his own estimation, exactly p times the value of the whole cake, where p is strictly greater than $1/n$, and, moreover, a best possible p can be attained. The number of distinct partitions of this kind, with the same optimal p, is either 1 or else greater than or equal to the power of the continuum. *P. R. Halmos* (Chicago, Ill.).

Urbanik, K. Sur un problème de J. F. Pál sur les courbes continues. Bull. Acad. Polon. Sci. Cl. III. 2, 205–207 (1954).

The problem referred to is solved by the following theorem. Let n be a positive integer and x a continuous mapping of the interval $0\leqq t\leqq 1$ in a metric space, and suppose $x(0)\neq x(1)$; then there exist $n+1$ values t_i of t such that $0=t_1<t_2<\cdots<t_{n+1}=1$ and the distances $\rho(x(t_{i-1}), x(t_i))$ are all equal $(i=1, 2, \cdots, n)$. Without the condition $x(0)\neq x(1)$ the theorem would be false. The proof is deduced from a form of the fixed-point theorem for an n-simplex. *A. H. Stone* (Manchester).

Lammel, Ernst. Über ein Rechnen mit reellen Zahlenfolgen. Arch. Math. 5, 385–388 (1954).

Given a real number q such that $0<q<1$, consider the collection S_q of all ordered pairs of double sequences $\{a_\mu\}$ $(\mu=0, \pm1, \pm2, \cdots)$ of real numbers for which the Laurent form $\sum_{\mu=-\infty}^{+\infty} a_\mu\rho^\mu$ in complex variable ρ defines a function regular in the circular ring $q<|\rho|<1/q$. The author introduces in S_q definitions of sum and product under which S_q forms an integral domain. *T. A. Botts.*

San Juan, Ricardo. A theorem on simply convergent improper integrals. Revista Mat. Hisp.-Amer. (4) 14, 74–75 (1954). (Spanish)

Let $\psi(x)$ be a non-decreasing function such that

$$\psi(x+\delta)/\psi(x)\leqq\phi(\delta)$$

for $x>x_0$, x_0 independent of δ, and let $\phi(\delta)=o(\delta^{-1})$ as $\delta\to 0+$. Let $f(x)$ be continuous for $x>x_0$ and let

$$-k\psi(x)\leqq\underline{f}'(x-)\leqq\bar{f}'(x-)\leqq k\psi(x),$$
$$-k\psi(x)\leqq\underline{f}'(x+)\leqq\bar{f}'(x+)\leqq k\psi(x),$$

for all $x>x_1$, except for a denumerably infinite set of values. Then the convergence of the integral $\int^\infty f(x)dx$ entrains $f(x)=o[\psi(x)]$ as $x\to+\infty$. *I. I. Hirschman, Jr.*

Gonçalves, J. Vicente. Sur la variation totale des fonctions discontinues. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) 3, 137–142 (1954).

A real-valued function f on a real interval $[a, b]$ is said to be moderate at the point c of $[a, b]$ provided for every $\epsilon>0$ there is a $\delta>0$ such that if $x'<x<x''$ are three points of $[a, b]$ in the δ-neighborhood of c, then

$$|f(x)-f(x')|+|f(x'')-f(x)|-|f(x'')-f(x')|<\epsilon.$$

It is shown that f is moderate at c if and only if the right and left limits $f(c-0)$ and $f(c+0)$ exist and are finite and $f(c)$ lies in the closed interval with endpoints $f(c-0)$ and $f(c+0)$. Also, if f is moderate at every point of $[a, b]$, then the variation $V_P(f)$ for a partition P of $[a, b]$ tends to the total variation (finite or infinite) of f on $[a, b]$ as the norm of P tends to zero; and if f is of bounded variation on $[a, b]$, the converse holds. *T. A. Botts* (Charlottesville, Va.).

Pini, Bruno. Sulle funzioni a variazione doppia limitata d'ordine maggiore di uno. Atti Sem. Mat. Fis. Univ. Modena 6 (1951–52), 34–44 (1953).

A function $f(x)$ is said to belong to the Wiener class W_p, $p\geqq 1$, on an interval $[a, b]$ when $\sum_{i=1}^n|f(x_i)-f(x_{i-1})|^p$ is bounded for $a=x_0<x_1<\cdots<x_n=b$; and an analogous class of functions $f(x, y)$ of two variables has been considered by L. C. Young [Math. Ann. 115, 581–612 (1938)]. The author considers alternative analogues that reduce for $p=1$ to the classes of functions $f(x, y)$ of bounded variation in the senses of Hardy-Krause and Arzelà [cf. J. A. Clarkson and C. R. Adams, Trans. Amer. Math. Soc. 35, 824–854 (1933); 36, 711–730 (1934)]. If r is a rectangle $[x', x''; y', y'']$, he writes $\Delta_r f(x, y)$ for $f(x'', y'')-f(x'', y')-f(x', y'')+f(x', y')$ and says that $f(x, y)$ belongs to the class $HK-p$ on $R[a, b; c, d]$ when (i) $\sum_r|\Delta_r f(x, y)|^p$ is bounded for all divisions of R into non-overlapping subrectangles r, (ii) $f(x, c)$ belongs to W_p on $[a, b]$, and similarly for $f(a, y)$. Further, he says that $f(x, y)$ belongs to $A-p$ on R when $\sum_{i=1}^n|f(x_i, y_i)-f(x_{i-1}, y_{i-1})|^p$ is bounded for

$$a=x_0<x_1<\cdots<x_n=b,\quad c=y_0<y_1<\cdots<y_n=d.$$

He proves that $HK-p\subseteq A-p$ and that if $f(x, y)$ is in $HK-p$ then the points where it is discontinuous with

respect to x have abscissae belonging to a finite or denumerable set, and similarly for y; these results were known for $p=1$ [cf. Clarkson and Adams, loc. cit.]. However, he shows that a function in $HK-p$ need not be totally differentiable almost everywhere, unless $p=1$.

H. P. Mulholland (Birmingham).

Vituškin, A. G. Determination of the variations of a set and the metric duality law. Doklady Akad. Nauk SSSR (N.S.) **96**, 893–896 (1954). (Russian)

Vituškin, A. G. Variations of a function of many variables and sufficient conditions for their boundedness. Doklady Akad. Nauk SSSR (N.S.) **96**, 1089–1091 (1954).

The author generalizes a procedure of a previous note [Vituškin, same Doklady (N.S.) **95**, 433–434 (1954); these Rev. **15**, 945], and defines, for $k=0, 1, \cdots, n$, a variation $V_k(E)$ for certain measurable sets E situated in Euclidean n-space and observes that these variations (for varying k) are independent. Given a set E situated in a cube Y of side $a\leqq 1$, he terms metric duality law a lower estimate, in terms of the $V_k(E)$, for the size of a parallel cube contained in $Y-E$. Given a real function f defined in a subset of Euclidean n-space, he defines $V_k(f)$ as the integral in t of $V_{k-1}(E_t)$, where E_t is the set in which f takes the value t; he observes that $V_{n+1}(f)$ is 0 and that the $V_k(f)$ for $k=1, \cdots, n$ are independent. His final theorem states that if f has partial derivatives of orders $\leqq n-k$ subject to a Lipschitz condition with constant L_1, then $V_k(f)$ does not exceed a constant depending only on L_1 and on n. Proofs are omitted and in some of the statements (Lemmas 5, 6, Theorems 5, 6 of the second paper) the author has neglected to explain his notation. (The symbols h and α' with suffixes i are not defined.) *L. C. Young* (Madison, Wis.).

Il'in, V. P. On a theorem of inclusion for a limiting exponent. Doklady Akad. Nauk SSSR (N.S.) **96**, 905–908 (1954). (Russian)

The author extends to the limiting case $q=mp/(n-lp)$ a result due to Sobolev and Kondrashov, according to which a function of n variables, possessing generalized partial derivatives of orders $\leqq l$ which are summable to the power $p>1$ in a domain D, is itself summable, on sections of D by hyperplanes of dimension $m>n-lp$, to the power q where $q<mp/(n-lp)$. *L. C. Young* (Madison, Wis.).

Kudryavcev, L. D. On implicit functions. Uspehi Matem. Nauk (N.S.) **9**, no. 3(61), 155–156 (1954). (Russian)

The author establishes briefly the following implicit-function theorem which extends a classical theorem of W. H. Young, where continuity of partial derivatives is likewise not assumed: Let $F(x, y)$ be a differentiable mapping, into Euclidean q-space E^q, of a domain G of E^{p+q}, and suppose that the Jacobian of F with respect to y of E^q does not vanish anywhere in G. Then the equation $F(x, y)=0$ defines, in a neighbourhood of each point (x_0, y_0) at which it is satisfied, a unique differentiable mapping $y=y(x)$ of a neighbourhood of x_0 into E^q. *L. C. Young.*

Dinghas, Alexandre. Sur une généralisation du théorème de Lusternik concernant des familles continues des ensembles. C. R. Acad. Sci. Paris **239**, 575–576 (1954).

Dinghas, Alexandre. Démonstration du théorème de Brunn-Minkowski pour des familles continues d'ensembles. C. R. Acad. Sci. Paris **239**, 605–607 (1954).

The author defines a linear metric space X whose elements are closed sets of Euclidean m-space, linear combination being understood as in Minkowski's theory of convex figures, and he considers integrals of the form $A^*=\int A(\lambda)d\lambda$ on the segment $0\leqq\lambda\leqq 1$ whose values are in X. The inequality $\varphi(A^*)\geqq\int\varphi[A(\lambda)]d\lambda$, where the nth power of φ is n-dimensional measure, is announced in the first note and its proof is sketched in the second note. A lengthy paper will deal with the details. *L. C. Young.*

Marstrand, J. M. Some fundamental geometrical properties of plane sets of fractional dimensions. Proc. London Math. Soc. (3) **4**, 257–302 (1954).

The author studies plane sets which are measurable with respect to Hausdorff s-dimensional measure ($0\leqq s\leqq 2$), and he terms such sets s-sets. He makes use of methods similar to those devised by Besicovitch in the case $s=1$, but while some of his results are analogous to Besicovitch's, others show considerable differences. He finds in particular that, for values of s other than 0, 1, 2, regular s-sets cannot exist and a much sharper angular density theorem holds. In this connection, the reviewer remarks that for $s=1$ a regular s-set is an s-set situated on countably many rectifiable curves: it might be interesting to ask what s-sets are situated on countably many s-continua. *L. C. Young.*

Steinhaus, H. Length, shape and area. Colloquium Math. **3**, 1–13 (1954).

The paper deals almost exclusively with length. The author recalls the probabilistic background of the integral-geometric formula for length given in Blaschke's book [Vorlesungen über Integralgeometrie, H. 1, 2. Aufl., Teubner, Leipzig-Berlin, 1936] and he makes suggestions for its further study. (Some of the questions raised are stated to have been solved subsequently by H. Fast.) In particular, the author defines a new distance function in the space of rectifiable curves and several new modes of convergence. He states that the paper is an invitation for the reader to solve the problems of pure and applied geometry involved in its approach to the notion of length and shape rather than an attempt of the author to answer the questions by himself. *L. C. Young* (Madison, Wis.).

Theory of Functions of Complex Variables

***Privalov, I. I. Vvedenie v teoriyu funkcil kompleksnogo peremennogo. [Introduction to the theory of functions of a complex variable.]** 9th ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1954. 444 pp. 10.40 rubles.

For a review of the 8th edition (1948) see these Rev. **13**, 334.

Ullman, J. L. On Faber series. I. A problem of transfer. Michigan Math. J. **2**, 109–114 (1954).

L'auteur démontre qu'une condition nécessaire et suffisante pour qu'une fonction $F(z)$ analytique à l'intérieur d'une courbe analytique C, et dont le développement en série de Faber est $\sum a_n F_n(z)$, soit rationnelle est que la fonction $\sum a_n t^n$ le soit. De là il tire le théorème d'Iliev, qui généralise aux sèries de Faber le théorème de Szegö, et généralise, de la même façon, le théorème de Carlson-Pólya.

S. Mandelbrojt (Paris).

Pandey, Nirmala. On the analytic continuation of certain series. Math. Student **22**, 95–100 (1954).

In this and the following paper the author employs methods introduced by Lindelöf [Le calcul des résidus et ses

applications à la théorie des fonctions, Gauthier-Villars, Paris, 1905, pp. 108–138] to a class of series to prove some theorems regarding the continuation of the function defined by the series to regions beyond the natural domain of convergence. The following theorem is typical. Let

$$H(s,z)=\sum_{n=1}^{\infty}\exp(Ain^{\beta}-sn^{\alpha})\,\Gamma(n)/\Gamma(z+n),$$
$$A>0,\quad 0<\alpha<\beta<1;$$

then $H(s,z)$ is an integral function of both z and s.

V. F. Cowling (Lexington, Ky.).

Pandey, Nirmala. On the analytic continuation of Newton series. Math. Student **22**, 89–93 (1954).

By methods similar to those employed in the paper reviewed above, the author proves the following theorem concerning the analytic continuation of function defined by Newton series. Let

$$\Omega(z)=\sum_{n=0}^{\infty}(-1)^n\exp(Ain)\,g(n)(z-1)(z-2)\cdots(z-n),$$
$$0<A<2\pi,$$

have a finite abscissa of convergence λ. Let $g(w)$ $w=x+iy$, satisfy the following conditions: (a) it is regular in some half-plane $x\geqq h$, the values of $g(k)$, $k=0,1,\cdots,[h]$, being finite numbers; (b) uniformly for $x\geqq h$,

$$\limsup_{|y|\to\infty}|y|^{-1}\log|[g(x+iy)/g(x)]|\leqq d,$$

where d is the smaller of the two numbers $A-\pi/2$ and $3\pi/2-A$. Then the function $\Omega(z)$ represents an integral function of z. A valid representation is also obtained.

V. F. Cowling (Lexington, Ky.).

Kubo, Tadao. Kelvin principle and some inequalities in the theory of functions. I. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **28**, 299–311 (1954).

The author develops a unified method, based on Kelvin's principle, for deducing various inequalities in the theory of functions. The method applies to inequalities which may be reduced to statements regarding the properties of harmonic functions with a vanishing normal derivative on some of the boundary components of the given domain. The results obtained concern Bergman kernel function and certain extremal properties of one-one conformal mappings onto various slit domains. *O. Lehto* (Helsinki).

Rahmanov, B. N. On the theory of univalent functions. Doklady Akad. Nauk SSSR (N.S.) **97**, 973–976 (1954). (Russian)

The author generalizes the concept of a starlike domain in six different ways and indicates the possibility of still further different generalizations. In three of the generalizations it is required that for each boundary point, M of D, a prescribed arc of a certain parabola (in place of the customary line to the origin) must lie in D. In the other generalizations, it is required that a segment of a line from M tangent to a certain parabola lies in D (or in the complement of D). The author announces six theorems (one per type of domain) giving necessary and sufficient conditions that a function $f(z)$, $f(0)=0$, $f'(0)=1$, regular in $|z|\leqq 1$ except for a finite number of boundary points, be univalent in $|z|\leqq 1$ and map this circle onto a domain of prescribed type. Each theorem is a generalization of the condition $\Re(zf'(z)/f(z))=0$, for a starlike domain. Extensions are given for mapping onto the complement of a domain of prescribed type. Only an outline of the method of proof is given. *A. W. Goodman* (Lexington, Ky.).

Kufarev, P. P. On a property of extremal regions of the problem of coefficients. Doklady Akad. Nauk SSSR (N.S.) **97**, 391–393 (1954). (Russian)

Let $f(w)=z+\sum_{n=2}^{\infty}c_nw^n$ be regular and univalent in $|w|<1$ and let B be the image domain for a function $f(w)$ for which $\Re c_n$ is a maximum. Let $C_{z_0}(\tau)$ be an arbitrary arc of the boundary of B joining z_0 and $z(\tau)$, let $G_{z_0}(\tau)$ be the domain obtained from the plane by deleting this arc, and let $w=F(z,\tau)$ map $G_{z_0}(\tau)$ univalently onto $|w|<1$, with $F(0,\tau)=0$, $F_z'(0,\tau)=e^{\tau}$. Then $F(z,\tau)$ satisfies a differential equation of the form $(zF^{-1}\partial F/\partial z)^2=A(z)/B(F,\tau)$, where $A(z)$ and $B(F,\tau)$, though complicated, are explicitly given.

A. W. Goodman (Lexington, Ky.).

Wilson, R. Note on a previous paper. Quart. J. Math., Oxford Ser. (2) **5**, 99–101 (1954).

The author generalizes one of his theorems on the coefficient theory of entire functions of finite order and mean type [same J. (2) **4**, 142–149 (1953); these Rev. **15**, 22].

R. P. Boas, Jr. (Evanston, Ill.).

Shah, S. M., and Singh, S. K. The maximum term of an entire series. Proc. Roy. Soc. Edinburgh. Sect. A. **64**, 80–89 (1954).

Let $f(z)$ be an entire function, $M(r)$ its maximum modulus, $\mu(r)$ the maximum term in its power series, and $\nu(r)$ the rank of the maximum term. Let d, c denote the lim sup and lim inf of $\{\log M(r)\}/\nu(r)$. If $f(z)$ is of infinite order, it is known that $c=0$ but that, for any $\psi(r)\uparrow\infty$, it is possible to have $\lim\psi(r)\{\log M(r)\}/\nu(r)=\infty$. The authors show that for a certain class of increasing functions $\theta(x)$ and for entire functions of infinite order whose coefficients satisfy certain conditions of regularity,

$$\lim\frac{\log M(r)\theta(\log M(r))}{\nu(r)}=0.$$

It is known that, if $f(z)$ is of zero order, $d=\infty$ while c may have any value. Given $\psi(x)\to\infty$ (slowly) and $\phi(x)\to\infty$ (rapidly), the authors construct entire functions of zero order with $\lim\{\log M(r)\}/\{\nu(r)\psi(\nu(r))\}=0$ or with

$$\lim\{\log M(r)\}/\phi(\nu(r))=\infty.$$

Finally, if $L(r)$ is positive and continuous, $L(r)\to\infty$, and $L(kr)\sim L(r)$, then $\log M(r)\sim L(r)$ or $\nu(r)\sim L(r)$ implies $c=\infty$; while if $l(r)=A(\log r)^{\alpha_1}(\log\log r)^{\alpha_2}\cdots$ is a product of powers of iterated logarithms, and $l(r)\to\infty$, then $\log M(r)\sim(\log r)l(r)$ implies $\nu(r)\sim(1+\alpha_1)l(r)$. There are some remarks on relations of the form

$$\lim\{\log M(R)/\nu(r)\}=\infty,\quad \lim\{\log M(r)/\nu(R)\}=0,$$

where $R=R(r)>r$. *R. P. Boas, Jr.* (Evanston, Ill.).

Hayman, W. K. On Nevanlinna's second theorem and extensions. Rend. Circ. Mat. Palermo (2) **2** (1953), 346–392 (1954).

Let $f(z)$ be meromorphic in $|z|\leqq R<\infty$ with $n(r,f)$ poles in $|z|<r$, multiplicity counted, and $\bar{n}(r,f)$ poles without

regard for multiplicity. Define

$$N(r,f)=\int_{1/r}^{r}n(t,f)\frac{dt}{t},\qquad \bar{N}(r,f)=\int_{1/r}^{r}(\bar{n}t,f)\frac{dt}{t},$$

$$m(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{i\theta})|\,d\theta,\quad T(r,f)=m(r,f)+N(r,f),$$

$$m(z_0,r,f(z))=m(r,f(z_0+z)),$$

with similar definitions for N, $\bar{N}$, T when the circles have variable centers z_0. For a sequence of circles $|z-z_n|<r_n$ define the defect

$$\Theta(a)=1-\limsup\frac{\bar{N}(z_n,r_n,(f-a)^{-1})}{T(z_n,r_n,f)}.$$

The author finds new conditions under which Nevanlinna's defect relation holds: (1) $\sum_{\nu=1}^{q}\Theta(a_\nu)\leqq 2$. It is found that (1) holds for functions of unbounded characteristic in $|z|<1$, which are meromorphic and of finite order in a larger domain, bounded by a finite number of analytic arcs lying in $|z|<1$ (except possibly for end joints).

If $f(z)$ is meromorphic of finite order in the plane and if $r_ne^{i\theta_n}$ are the roots of $f(z)=a$ in $|\arg z|<\pi/2\rho$, $\frac{1}{2}<\rho<\infty$, then either (i) $f(z)$ has bounded characteristic in $|\arg z|<\pi/2\rho$ and (2) $\sum\cos\rho\theta_n r_n^{-\rho}$ converges for every a; or (ii) $f(z)$ has unbounded characteristic in $|\arg z|<\pi/2\rho$ and the series (2) diverges for every a with at most two exceptions. If (2) diverges for one value a, then $f(z)$ takes every value with at most two exceptions infinitely often in the angle.

The author also finds that if the circles $|z-z_n|<r_n$ exhaust $|z|<1$ in the sense that

$$T(z_n,r_n,f)/\log((1-r_n)^{-1})\to\infty,$$

then (1) again holds. The Riemann surface is then said to be properly exhausted. This is shown to be the case whenever

$$(1-r)\log\sup_{|z|=r}|f(z)|\to\infty\quad\text{as}\quad r\to 1,$$

and this condition cannot be weakened.

Corresponding theorems are also obtained relating to the Ahlfors exhaustion theory. *M. S. Robertson.*

Gerstenhaber, Murray. A characterization of the modular group and certain similar groups. Duke Math. J. **21**, 113–121 (1954).

The following theorem is proved. Any two Fuchsian groups which are isomorphic to the free product of a cyclic group of order 2 and one of given order n ($\geqq 3$) and which have fundamental domains with finite hyperbolic areas are congruent in the sense of hyperbolic geometry, that is, they are conjugate within the full group of hyperbolic motions. The proof is based on an investigation of the Riemann surface determined by the commutator subgroup of such a group G_n. It is shown that this surface is uniquely determined by the genus and the symmetry properties which it must have because of the structure of G_n.

Reviewer's remark: The problem solved by the author and a more general one formulated at the end of the paper may be solved in a simple manner using the canonical representation and the theory of modules for Fuchsian groups [cf., e.g., R. Fricke, Encykl. Math. Wiss., Band II, 2. Teil, Teubner, Leipzig, 1913, pp. 349–470, in particular, Sections 8–10]. The problem in question is to characterize the groups which are realizable in one and only one way as Fuchsian groups with fundamental domains having finite hyperbolic areas. The structures of all Fuchsian groups being known, the problem reduces to picking out those among them which have no modules and which are not isomorphic to Fuchsian groups of a different type. The groups having no modules are precisely those generated by three elements σ_1, σ_2, σ_3 satisfying the relations $\sigma_1\sigma_2\sigma_3=1$, $\sigma_1^{n_1}=\sigma_2^{n_2}=\sigma_3^{n_3}=1$, where n_1, n_2, n_3 are positive integers or ∞ and $1/n_1+1/n_2+1/n_3<1$, and where σ_i is parabolic when $n_i=\infty$. Geometrically such a group is a subgroup of index 2 in the group generated by the reflections in the sides of the hyperbolic triangle with angles π/n_1, π/n_2, π/n_3. The only one of these groups which is isomorphic to a Fuchsian group of different type is that corresponding to $n_1=n_2=n_3=\infty$. *W. Fenchel.*

Lelong, Pierre. Sur l'étude des noyaux primaires et sur un théorème de divisibilité des fonctions entières de n variables. C. R. Acad. Sci. Paris **237**, 1379–1381 (1953).

Let $\{U_i, f_i\}$ be a Cousin-II-distribution in a domain $D\subset C^n$ (that is: $f_if_j^{-1}$ is holomorphic in $U_i\cap U_j$). Let

$$\omega_i=\sum_{\mu,\nu}\frac{\partial^2\log|f_i(z)|}{\partial z_\mu\partial\bar{z}_\nu}dz_\mu\wedge d\bar{z}_\nu.$$

Then we have $\omega_i=\omega_j$ in $U_i\cap U_j$. Therefore $\omega=\omega_i$ defines a current in D. Let $d\mu$ be the Radon measure associated with ω [cf. P. Lelong, same C. R. **237**, 691–693 (1953); these Rev. **15**, 415]. Let in the following $D=C^n$. Let $\mu(t)$ be the mean value of $d\mu$ on the sphere $|z|=t$. Then for $n\geqq 2$ $\nu(t)=(2n-2)\mu(t)t^{2n-3}$ is defined as "indicatrix." If $\limsup_{t\to\infty}\log\nu(t)/\log t=\lambda$, then the manifold W^{n-1} which is defined by the Cousin distribution is said to be "of the order λ." If W^{n-1} does not contain the origin and if W^{n-1} is of finite order λ, then there exists an entire function $F(z)$ of order λ, such that $W^{n-1}=\{z|F(z)=0\}$. $F(z)$ is given by an integral representation in which the kernels are used which P. Lelong has treated [ibid. **237**, 865–867 (1953); these Rev. **15**, 416]. Every entire function that vanishes on W^{n-1} is divisible by $F(z)$. W^{n-1} is algebraic if and only if the indicatrix $\nu(t)$ is bounded. *H. J. Bremermann.*

Lelong, Pierre. Sur les dérivées d'une fonction plurisousharmonique. C. R. Acad. Sci. Paris **238**, 2276–2278 (1954).

Let in the space C^n (space of n complex variables)

$$V^{i\bar{\jmath}}=\partial^2V/\partial z_i\partial\bar{z}_j\quad\text{and}\quad\delta[V,\alpha]=\sum_{i,j}V^{i\bar{\jmath}}\alpha_i\bar{\alpha}_j,$$

where $|\alpha|=1$. A real-valued function V is plurisubharmonic in a domain D if and only if (1) V is summable on every compact subset of D, (2) $\delta[V,\alpha]$, defined as distribution in D, is positive, (3) for all points $M\in D$: $V(M)=V_m(M)$, where $V_m(M)$ is the maximum of V at the point M. Then $\mu_{i\bar{\jmath}}(f)=V^{i\bar{\jmath}}dz_i\wedge d\bar{z}_j(f)$ is a Radon measure. We have $\mu_{k\bar{k}}\geqq 0$ and $\mu_{i\bar{\jmath}}=\bar{\mu}_{j\bar{\imath}}$. Let $\mu=\sum_k\mu_{k\bar{k}}$; then $|\mu_{i\bar{\jmath}}|_D\leqq|\mu|_D$ (where $|\ |_D$ is a certain norm). The author proceeds to define certain classes of exterior differential forms, in particular currents which are locally of the form $dzd\bar{z}V$ (V plurisubharmonic). Let W^p be an analytic set in D of pure complex dimension p. Let $\omega^{(\lambda)}$ be a covering. In each $\omega^{(\lambda)}$, W^p can be represented as the intersection of the zero manifolds of $n-p$ functions $f_\nu^{(\lambda)}$ holomorphic in $\omega^{(\lambda)}$. Let $V_\nu^{(\lambda)}=\log|f_\nu^{(\lambda)}|$; then the author defines the current

$$\theta=(i/\pi)^{n-p}dzd\bar{z}V_1^{(\lambda)}\cdots dzd\bar{z}V_{n-p}^{(\lambda)}.$$

For the integral over W^p of a p-form φ with compact support in D we have $\theta(\varphi)=\int_{W^p}\varphi$. The "surface element" $d\sigma$ of W^p

defined as a positive Radon measure is $d\sigma=[p!]^{-1}\theta\wedge\beta^p$, where $\beta=\frac{1}{2}dxdz|z|^2$. The area $\sigma(R)$ of W^p contained in a sphere of radius R divided by R^{2p} is a nondecreasing function of R and has a limit for $R\to 0$. *H. J. Bremermann.*

Theory of Series

Ostrowski, Alexander. Mathematische Miszellen. XXII. Über gewisse Ungleichungen zwischen monotonen Zahlenfolgen. Jber. Deutsch. Math. Verein. **57**, Abt. 1, 85–89 (1954).

Let $\{x_k\}$ and $\{y_k\}$ be two nondecreasing sets of n real numbers each. Let the distance between every two consecutive x_k's or between every two consecutive y_k's be at most 2ϵ. For each x_k let there be some y_l such that $|x_k-y_l|<\epsilon$, and for each y_m let there be some x_p such that $|x_p-y_m|<\epsilon$. Then, for all k, $|y_k-x_k|\leqq\epsilon u$, where u is the largest odd number not exceeding n; and this u is the best possible. *R. P. Boas, Jr.* (Evanston, Ill.).

Bosanquet, L. S. On convergence and summability factors in a sequence. Mathematika **1**, 24–44 (1954).

For the case in which k and r are nonnegative integers and p and q are real, the author gives a complete solution of a factor-sequence problem involving Cesàro transforms. Conditions on a sequence ϵ_n are given which are necessary and sufficient to ensure existence of constants $\beta_1, \beta_2, \cdots, \beta_r$ such that the C_r transform of the sequence $\epsilon_n s_n$ has the form

$$\sum_{j=1}^{r}\beta_j\binom{n+r-j}{n}+o(n^{r+p+q})$$

whenever s_n is a sequence whose C_k transform has the form

$$\sum_{j=1}^{k}\alpha_j\binom{n+k-j}{n}+o(n^{k+p}).$$

The conditions depend upon the given constants in ways which are quite simple except for the fact that when $r\geqq 1$ and $p+q=-1, -2, \cdots, -r$ the conditions are different and more stringent than they are in other cases. Many references to previously known special cases and related results are given. *R. P. Agnew* (Ithaca, N. Y.).

Padmavally, K. On the Cesàro summability of a class of functions. J. Indian Math. Soc. (N.S.) **17** (1953), 151–158 (1954).

If $F_1, F_2, \cdots, F_k$ are successive indefinite integrals of $f(x)$, and $\int_0^\infty F_k(x)a^{(k)}(x)dx$ converges, then the question whether $\int_0^\infty f(x)a(x)$ is summable (C, k) turns on whether $\sum_0^{k-1}(-1)^\nu F_{\nu+1}(x)a^{(\nu)}(x)$ tends to a limit (C, k) as $x\to\infty$. The author shows (Theorem II) that: if $F_{k+1}(x)=O(1)$, then a sufficient extra condition is $a^{(k)}(x)\to 0$ as $x\to\infty$. This completes and extends a result stated incorrectly by Rajagopal for a periodic $f(x)$ [same J. (N.S.) **11**, 22–27 (1947); these Rev. **9**, 425]. A series analogue extends a theorem of Macphail [Bull. Amer. Math. Soc. **47**, 483–487 (1941); these Rev. **3**, 148].

The reviewer remarks that the following extension of the author's Theorem I (on which Theorem II depends) may be formulated: If ρ is a non-negative integer, p, q are real, $p-q>-1$, and $G_\rho(x)$ is a ρth integral of $g(x)$ such that $G_\rho(x)=O(x^{1+p})$, then $g(x)b(x)=o(x^{p-q})$ (C, ρ), if $b(x)=o(x^{-q})$ and $b^{(\rho)}(x)=o(x^{-q-\rho})$. This includes a result of the reviewer in which all the G_ν vanish at the origin [J. London Math. Soc. **23**, 35–38 (1948); for a series analogue see Proc. London Math. Soc. (2) **50**, 295–304 (1948) [these Rev. 10, 112] and the paper reviewed above]. The author's Theorem II is a corollary of the case $p=q=-\rho$ (his Theorem I), with $\rho=k-\nu$, $g(x)=F_{\nu+1}(x)$, $b(x)=a^{(\nu)}(x)$, and $\nu=0, 1, \cdots, k-1$. *L. S. Bosanquet* (London).

Pati, T. Products of summability methods. Proc. Nat. Inst. Sci. India **20**, 348–351 (1954).

It is shown that if s_n is a sequence that is evaluable to s by the Abel method A and if t_n is the transform of s_n by a regular Hausdorff transformation H, then the sequence t_n is evaluable A to s. In the familiar algebra of methods for evaluation of sequences, the result can be put in the form $AH\supset A$. *R. P. Agnew* (Ithaca, N. Y.).

Kuttner, B. The problem of "total translativity" for Hölder summability. J. London Math. Soc. **29**, 486–491 (1954).

A method A is totally translative if A summability of a real series (1) $a_0+a_1+a_2+\cdots$ to the extended-real s, $-\infty\leqq s\leqq+\infty$, is equivalent to A summability of the series (2) $0+a_0+a_1+\cdots$ to s. It is shown here that for the Hölder method of order k, $0<k<1$, (H, k) summability of (1) to $-\infty\leqq s\leqq+\infty$ implies (H, k) summability of (2) to s; that for $-1<k<0$ or $k>1$, summability of (2) implies that of (1), and that the inverse statements are false. *G. G. Lorentz* (Detroit, Mich.).

Gaier, D., und Zeller, K. Über den O-Umkehrsatz für das C_k-Verfahren. Rend. Circ. Mat. Palermo (2) **3**, 83–88 (1954).

The authors give and discuss two proofs of the classic theorem to which the title refers. *R. P. Agnew.*

Peyerimhoff, Alexander. Über das Anwachsen der C_r-Mittel von Laplace-Integralen auf vertikalen Geraden. Math. Ann. **128**, 138–143 (1954).

Let β_k be the C_k summability axis of the Laplace-Stieltjes transform

$$\int_0^\infty e^{-st}d\alpha(t)=f(s)\quad(k\geqq 0);$$

if $\sigma>\beta_k$, then $f(\sigma+i\tau)=O(\tau^{k+1})$, $\tau\to\pm\infty$. The author shows that this result is best possible; i.e. τ^{k+1} cannot be replaced by $\tau^{k+1}/\eta(\tau)$ where $\eta(\tau)\to\infty$ as $\tau\to\pm\infty$. Moreover not only is this true for Laplace-Stieltjes transforms but for Laplace transforms and Dirichlet series as well. *I. I. Hirschman, Jr.* (St. Louis, Mo.).

Rajagopal, C. T. A generalization of Tauber's theorem and some Tauberian constants. II. Math. Z. **60**, 142–147 (1954).

[For part I see Math. Z. **57**, 405–414 (1953); these Rev. **14**, 958.] Let

$$\int_0^\infty N(x)dx=1,\quad\int_0^\infty|N(x)\log x|dx<\infty$$

and put

$$K(u)=\int_u^\infty N(x)dx.$$

Let $a_n=O(1)$. Let $0<\lambda_1<\lambda_2<\cdots$ and let $\liminf\lambda_{n+1}/\lambda_n=L>1$. Let x be related to n so that the numbers α and β defined by $\alpha=\limsup\lambda_n/x$ and $\beta=\liminf\lambda_{n+1}/x$ are finite

and positive and $\beta \leqq \alpha L$. Then

$$\limsup_{n\to\infty}\left|\sum_{p=1}^{\infty} a_p K\left(\frac{\lambda_p}{x}\right)-\sum_{p\leqq n} a_p\right| \leqq C \limsup_{n\to\infty}|a_n|,$$

where C is the constant defined by

$$C=\int_0^{\alpha}|N(u)|du+\int_{\beta}^{\infty}|N(u)|du+\frac{T(\alpha,\beta)}{\log L}$$

and

$$T(\alpha,\beta)=\int_0^{\alpha}|N(u)|\log\frac{\alpha}{u}du+\int_{\beta}^{\infty}|N(u)|\log\frac{u}{\beta}du.$$

R. P. Agnew (Ithaca, N. Y.).

Rajagopal, C. T. On an absolute constant in the theory of Tauberian series. II. Proc. Indian Acad. Sci. Sect. A. **39**, 272–281 (1954).

[For part I see same Proc. **28**, 537–544 (1948); **31**, 60–61 (1950); these Rev. **10**, 447; **12**, 21.] Let

$$\psi(u)\geqq 0,\quad \int_0^{\infty}\psi(u)|\log u|du<\infty,\quad \int_0^{\infty}\psi(u)du=1$$

and put $\phi(u)=\int_u^{\infty}\psi(x)dx$. Suppose there is a unique positive a_0 for which $\phi(a_0)=\frac{1}{2}$. Then the function $\tau(a)$ defined by

$$\tau(a)=\int_0^{\infty}\psi(u)|\log(u/a)|du=\int_0^{a}\frac{1-\phi(x)}{x}dx+\int_a^{\infty}\frac{\phi(x)}{x}dx$$

has a unique absolute minimum when $a=a_0$. The main result of the paper is that $\tau(a_0)$ is the least constant with the following property. If $0<\lambda_1<\lambda_2<\cdots$, $\lambda_n\to\infty$, $\lambda_n-\lambda_{n-1}=O(1)$ and if $\sum a_n$ is a series for which

$$\limsup_{n\to\infty}\frac{|a_n|\lambda_n}{\lambda_n-\lambda_{n-1}}=L<\infty,$$

then to each limit point s' of the sequence of partial sums of the series $\sum a_n$ corresponds a limit point s'' (t tending to 0) of the transform

$$\Phi(t)=\sum a_n\phi(\lambda_n t)$$

such that $|s'-s''|\leqq\tau(a_0)L$. *R. P. Agnew.*

Gonçalves, J. Vicente. Un raffinement des séries de Pringsheim. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) **3**, 197–199 (1954).

Let $M_1, M_2, \cdots$ be an increasing sequence of positive numbers for which $M_n\to\infty$. Starting with the simple fact that $\sum(M_{n+1}-M_n)/M_nM_{n+1}$ diverges when $\rho=0$ and converges when $\rho=1$, one can apply the result to M_n^{ρ} and use elementary inequalities to show that the series converges when $\rho>0$. Applying these results to $\log M_n$ and using only elementary inequalities, the author shows very simply that the series

$$\sum(M_{n+1}-M_n)/M_n\log M_{n+1}$$

diverges and that the series

$$\sum(M_{n+1}-M_n)/M_{n+1}(\log M_{n+1})(\log M_n)^{\rho}$$

converges when $\rho>0$. Repetition of the procedure shows that the series

$$\sum(M_{n+1}-M_n)/M_{n+1}(\log M_{n+1})(\log\log M_{n+1})(\log\log M_n)^{\rho}$$

converges when $\rho>0$, and so on. *R. P. Agnew.*

Lafleur, Charles. Sur un développement en série de Taylor de la fonction de Dirac, et un développement en série de fonctions de Dirac de la fonction e^{ixz}. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **40**, 784–786 (1954).

The author finds formal developments of a Taylor series of a function and its moments in terms of the Dirac delta function. *A. Heins* (Pittsburgh, Pa.).

Brauer, George. Sets of convergence of ordinary Dirichlet series. Duke Math. J. **21**, 593–594 (1954).

"If α is any real number, and M is a set of type F_σ on the line L: $\sigma=\alpha$, where $s=\sigma+it$, then there exists an ordinary Dirichlet series $\sum_{n=1}^{\infty}a_n n^{-s}$ which converges on M and diverges on $L-M$." Sets of convergence for power series had been investigated previously by Herzog and Piranian [same J. **16**, 529–534 (1949); **20**, 41–54 (1953); these Rev. **11**, 91; **14**, 738]. *D. Gaier* (Stuttgart).

Tanaka, Chuji. Note on Dirichlet series. XII. On the analogy between singularities and order-directions. I. Proc. Japan Acad. **30**, 157–159 (1954).

Tanaka, Chuji. Note on Dirichlet series. XIII. On the analogy between singularities and order-directions. II. Proc. Japan Acad. **30**, 257–261 (1954).

These notes connect with Note XI of the same series [same Proc. **29**, 478–481 (1953); these Rev. **15**, 951] where the notion of order curve was introduced. An order direction is a horizontal order curve, $\Im(s)=t$, such that for every $\epsilon>0$ the entire function $F(s)=\sum_1^{\infty}a_n\exp(-\lambda_n s)$, where the series converges for all s, has the same order in $|\Im(s)-t|<\epsilon$ as in the whole plane. In the first note the author proves that if

$$\Re(a_n)\geqq 0\quad\text{and}\quad\lim_{n\to\infty}(\lambda_n\log\lambda_n)^{-1}\log\cos\arg a_n=0,$$

then $\Im(s)=0$ is an order direction. The analogous theorem for singularities due to C. Biggeri [C. R. Acad. Sci. Paris **209**, 979–980 (1939); these Rev. **1**, 113] asserts that $F(s)$ has $s=0$ as a singular point if $\Re(a_n)\geqq 0$ and

$$\lim_{n\to\infty}\lambda_n^{-1}\log\cos\arg a_n=0$$

when the series has $\sigma_0=0$ as abscissa of ordinary convergence. In the second note the condition on $\arg a_n$ is kept but $\Re(a_n)$ is allowed to change sign, say between $n=p_\nu$ and $p_\nu+1$ where

$$\liminf[\lambda_{p_\nu+1}-\lambda_{p_\nu}]=g>0,\quad\limsup\nu/r_\nu=\delta(\leqq 1/g)$$

and $r_\nu=\frac{1}{2}(\lambda_{p_\nu+1}+\lambda_{p_\nu})$. Then there is at least one order direction in the interval $|\Im(s)|\leqq\pi\delta$. If $\liminf(\lambda_{n+1}-\lambda_n)>0$ and $|\arg a_n|\leqq\theta<\frac{1}{2}\pi$ except for $n=n_k$ where $k/\lambda_{n_k}\to 0$, then $\Im(s)=0$ is an order direction. *E. Hille.*

Fourier Series and Generalizations, Integral Transforms

Men'šov, D. E. On some properties of Fourier series. Izvestiya Akad. Nauk SSSR. Ser. Mat. **18**, 379–388 (1954). (Russian)

A class K of functions integrable over $[-\pi,\pi]$ is said to be of type (α) if whenever $f(x)\in K$ and $f_0(x)$ is measurable, the inequality $|f_0(x)|\leqq|f(x)|$ implies that $f_0(x)\in K$ (for example, the classes $L^p(-\pi,\pi)$ are of type α). The author shows that each $f\in K$ can be represented in the form $f=f_1+f_2$, where f_1 and f_2 are in K, the Fourier series of f_i $(i=1,2)$ converges to 0 on a set $E_i\subset[-\pi,\pi]$ with

meas$(E_1+E_2)=2\pi$, and meas$(E_i\cdot[a,b])>0$ for each subinterval $[a,b]$ of $[-\pi,\pi]$. *A. Zygmund* (Chicago, Ill.).

Noble, M. E. Coefficient properties of Fourier series with a gap condition. Math. Ann. **128**, 55–62; correction, 256 (1954).

The author considers the Fourier series

$$(*)\qquad f(x)\sim\sum_{k=1}^{\infty}(a_{n_k}\cos n_kx+b_{n_k}\sin n_kx)$$

where $\min(n_{k+1}-n_k, n_k-n_{k-1})/\log n_k\to\infty$ $(k\to\infty)$. This is a weaker gap condition than Hadamard's. For such series he proves the following theorems. (1) If $f(x)$ is of bounded variation in a small interval, then

$$a_{n_k}=O(1/n_k),\quad b_{n_k}=O(1/n_k).$$

(2) If $f(x)$ belongs to the Lip α class $(0<\alpha<1)$ in a small interval, then $a_{n_k}=O(1/n_k^{\alpha})$, $b_{n_k}=O(1/n_k^{\alpha})$. (3) If $f(x)$ belongs to the Lip α class $(\frac{1}{2}<\alpha<1)$ in a small interval, then the series (*) converges absolutely. (4) If $f(x)$ is of bounded variation and belongs to the Lip α class $(0<\alpha<1)$ in a small interval, then the series (*) converges absolutely. Proof of these theorems depends on use of the formula

$$a_{n_k}=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)P(x)\cos n_kx\,dx$$

and similar one for b_{n_k}, where $P(x)$ is a trigonometric polynomial with constant term 1 and with degree less than $\min(n_{k+1}-n_k, n_k-n_{k-1})$. *S. Izumi* (Tokyo).

Mohanty, R., and Nanda, M. Note on the first Cesàro mean of the derived Fourier series. Proc. Amer. Math. Soc. **5**, 566–570 (1954).

Let $f(t)$ be an integrable function in $(-\pi,\pi)$, with period 2π, and let $s_n'(x)$ and $\sigma_n'(x)$ be the nth partial sum and its nth arithmetic mean of the differentiated Fourier series of $f(x)$. The authors prove that if

$$\int_0^t\left|\frac{f(x+u)-f(x-u)}{u}-c\right|du=o(t)\quad\text{as}\quad t\to0,$$

then $s_n'(x)=o(n)$ and $\sigma_n'(x)=o(\log n)$, and $1/\log(n+1)$ is the $(C,1)$ summability factor of the differentiated Fourier series. *S. Izumi* (Tokyo).

Žak, I. E. On a theorem of Zygmund's on conjugate series. Doklady Akad. Nauk SSSR (N.S.) **97**, 387–389 (1954). (Russian)

Let $f(x,y)$ be continuous in (x,y) and of period 2π in x and y. It is said to belong to the class Lip α, if

$$|f(x+h,y+k)-f(x)|\leqq C(|h|^{\alpha}+|k|^{\alpha}).$$

In a previous paper [same Doklady (N.S.) **87**, 877–880 (1952); these Rev. **14**, 975, 1279] the author shows that, if $f\in$ Lip α, the three conjugates one can associate with f need not be in Lip α (though they are in Lip α' for each $\alpha'<\alpha$). The author now shows that, if $f(x,y)$ satisfies the condition

$$f(x+h,y+\eta)+f(x-h,y-\eta)-2f(x,y)=O(|h|)+O(|\eta|)$$

(which is a generalization of the condition Lip 1), the conjugate functions need not satisfy it. *A. Zygmund.*

Nikolaev, V. F. On a class of polynomial operations. Doklady Akad. Nauk SSSR (N.S.) **96**, 17–19 (1954). (Russian)

Let C be the linear space of continuous functions of period 2π with the maximum as norm. Let $U_{n,m}$ be an operator which carries each element of C into a trigonometric polynomial of order at most $n+m-1$, while it leaves trigonometric polynomials of order n (or less) invariant. The problem of determining the minimum norm of $U_{n,m}$ has been solved only when $m=1$ [Lozinskiĭ, same Doklady (N.S.) **61**, 193–196 (1948); these Rev. **10**, 188], when the minimum norm is attained for the operator carrying each function into the nth partial sum S_n of its Fourier series. The author [same Doklady **68**, 9–11 (1949); these Rev. **11**, 100] showed that $\|U_{n,m}\|>A\log\{(n+m-1)/m\}$. Here he shows that if m divides n, the minimum norm is attained for the operator which carries a function into

$$m^{-1}(S_n+S_{n+1}+\cdots+S_{n+m-1}),$$

so that the minimum norm is

$$2\pi^{-1}\int_0^{\pi}x^{-2}|\sin\{(2n+m)/m\}x\cdot\sin x|\,dx.$$

R. P. Boas, Jr. (Evanston, Ill.).

Littlewood, J. E. On a theorem of Paley. J. London Math. Soc. **29**, 387–395 (1954).

The note is concerned with the following result of Paley [Studia Math. **3**, 226–238 (1931)]. Let $\phi_1(x),\phi_2(x),\cdots,\phi_n(x),\cdots$ be a complete orthonormal system over $(0,1)$, and let $|\phi_n|<M$ for all n. Then, if $f\sim\sum c_n\phi_n$, we have

$$(*)\qquad\left(\int_0^1|\phi|^q\,dx\right)^{1/q}\leqq A_qM^{(q-2)/q}\left(\sum|c_n|^q n^{q-2}\right)^{1/q}\quad(q\geqq2)$$

where A_q depends on q only. Unlike the familiar Hausdorff-Young inequality

$$\left(\int_0^1|\phi|^q\,dx\right)^{1/q}\leqq M^{(q-2)/q}\left(\sum|c_n|^p\right)^{1/p},\quad 1/p+1/q=1,$$

(*) cannot be obtained by interpolation between $q=2$ and $q=\infty$ by means of M. Riesz' method [Acta Math. **49**, 465–497 (1927)]. The purpose of this note is to show that for all $2\leqq q<\infty$ we have

$$(**)\qquad\left(\int_0^1|\phi|^q\,dx\right)^{1/q}\leqq A\left(\sum|c_n|^2C_n^{q-2}\right)^{1/q},$$

where $C_n=|c_1|+\cdots+|c_n|$, and A is an absolute constant. From this (*) follows easily (with $A_q=Aq$).

A. Zygmund (Chicago, Ill.).

Andreoli, Giulio. Su due sistemi di funzioni ortogonali costanti a tratti e collegati a determinanti di Hadamard. Ricerca, Napoli **5**, no. 1–2, 3–14 (1954).

Construction de deux systèmes orthogonaux complets dans $0<x<1$. Partant de $\delta(x)=1$ pour $0<x<1$, $\delta(x)=-1$ pour $1<x<2$, et $\delta(x)$ de période 2, et écrivant pour k entier $\geqq0$: $k=2^{\rho}+2^{\sigma}+\cdots$ $(\rho>\sigma>\cdots)$, le premier système est défini par la suite: $w_k(x)=\delta(2^{\rho}x)\cdot\delta(2^{\sigma}x)\cdots$. Le deuxième système ne me paraît pas différent de celui de Haar.

J. Favard (Paris).

Kawata, Tatsuo. A theorem on Fourier transform. Kōdai Math. Sem. Rep. **1954**, 22–24 (1954).

An attempt is made to generalise the classical result due to M. Plancherel: if $f(x)\in L_2(-\infty,\infty)$, then there is a function $g(\theta)\in L_2(-\infty,\infty)$, the Fourier transform of $f(x)$, such that

$$\int_{-\infty}^{\infty}\left|f(x)-\int_{-A}^{A}e^{-ix\theta}g(\theta)d\theta\right|^2dx\to0\quad\text{as}\quad A\to\infty.$$

Here the following problem is treated: let $F(x)$ be a fixed

non-decreasing function, bounded in $(-\infty, \infty)$. If

(i) $$\int_{-\infty}^{\infty} |f(x)|^2 dF(x) < \infty,$$

is there a function $K(\theta)$ of bounded variation such that

(1) $$\int_{-\infty}^{\infty} \left| f(x) - \int_{-A}^{A} e^{-ix\theta} dK(\theta) \right|^2 dF(x) \to 0 \quad \text{as} \quad A \to \infty ?$$

The author proves: if $f(x)$ satisfies (i), and

(ii) $$\int_{-\infty}^{\infty} |f(x+u) - f(x)|^2 dF(x) \to 0 \quad (u \to 0),$$

and (iii) $f(x) \in L_2(-A, A)$ for any $A > 0$, then, given $\epsilon > 0$, there exist an A and a function $K(\theta) = K_A(\theta)$ of bounded variation in $(-A, A)$ and depending on A, such that the term on the left of (1) is $< \epsilon$. In fact, his function $K_A(\theta)$ is absolutely continuous. *H. Kober* (Birmingham).

Malgrange, Bernard. Sur quelques propriétés des équations de convolution. C. R. Acad. Sci. Paris **238**, 2219–2221 (1954).

The author announces results which generalize some of the results of L. Schwartz [Ann. of Math. (2) **48**, 857–929 (1947); these Rev. **9**, 428] dealing with mean periodic analytic functions from one to several variables. Let H be the space of entire functions of $z = (z_1, z_2, \cdots, z_n)$ and let H' be the dual space. If $\mu \in H'$, let $F(\mu)$ be its Fourier transform; for any $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$, $F(\mu)(\lambda) = \langle \mu, f \rangle$ where $f(z) = \exp(-2\pi i[\lambda_1 z_1 + \cdots + \lambda_n z_n])$. This is an entire function of λ, and every such function can be obtained in this way. As in Schwartz's paper, one considers the convolution $\mu * \nu$ of two functionals, and the mapping $f \to \mu * f$ of H into itself, determined by $\mu \in H'$. Given μ and ν, a necessary and sufficient condition that there be a σ such that $\nu = \mu * \sigma$ is that $F(\nu)/F(\mu)$ be entire. Let V be the subspace of H consisting of all f with $\mu * f = 0$ and E the set of exponential polynomials which it contains; then E is dense in V. Analogous results are stated for more general spaces of functions. *R. C. Buck* (Madison, Wis.).

Herz, Carl S. On the mean inversion of Fourier and Hankel transforms. Proc. Nat. Acad. Sci. U. S. A. **40**, 996–999 (1954).

Let t denote a point of Euclidean k-dimensional space R_k. A function $F(t) \in L^p(R_k)$, $1 \leq p \leq 2$, has a Fourier transform $G(s) = \int_{R_k} \exp(is \cdot t) F(t) dt$. Let C denote any convex polyhedron in R_k containing the origin, and nC $(n > 0)$ its dilations. If

(1) $$F_n(t) = (2\pi)^{-k} \int_{nC} \exp(-is \cdot t) G(s) ds,$$

then for $1 < p < 2$

(2) $$\|F_n\|_p \leq A(p) \|F\|_p, \quad \|F_n - F\|_p \to 0 \quad \text{as} \quad n \to \infty.$$

The demonstration of this result is an easy consequence of M. Riesz's theorem on conjugate functions. It is pointed out that if the polygon C is replaced by a sphere then (2) is false for $1 \leq p \leq 2k/(k+1)$. While the author has not been able to prove that (2) holds in general for $2k/(k+1) < p \leq 2$ he has been able to verify it in certain special cases, notably when $F(t)$ is of the form $f(t \cdot t) P(t)/|t|^\nu$, where $P(t)$ is a harmonic polynomial homogeneous of degree ν. This is deduced from the following result. Let $A_k(x) = J_k(2\sqrt{x}) x^{-k/2}$ and let

$$g(y) = \int_0^\infty A_{\gamma+\nu}(xy)(xy)^{\nu/2} f(x) x^\gamma dx$$

where $\gamma \geq -\frac{1}{2}$, $\nu \geq 0$. If

$$f_n(x) = \int_0^\infty A_{\gamma+\nu}(xy)(xy)^{\nu/2} g(y) y^\gamma dy$$

then

$$\int_0^\infty |f_n(x)|^p x^\gamma dx \leq B(p, \gamma) \int_0^\infty |f(x)|^p x^\gamma dx.$$

The demonstration of this follows ideas of Pollard and Wing [see, e.g., Wing, Amer. J. Math. **72**, 792–808 (1950); these Rev. **12**, 329]. *I. I. Hirschman, Jr.* (St. Louis, Mo.).

Fox, Charles. Chain transforms. Proc. Amer. Math. Soc. **5**, 677–688 (1954).

The author considers a chain transform of order n in generalising chains of order 3 [see C. Fox, J. London Math. Soc. **23**, 229–235 (1948); these Rev. **10**, 371]. Such transforms are defined by the equations

$$g_{p+1}(x) = \int_0^\infty r_p\left(\frac{x}{u}\right) g_p(u) \frac{du}{u}, \quad \int_0^\infty l_q\left(\frac{x}{u}\right) g_{q+1}(u) \frac{du}{u} = g_q(x),$$

where $p, q = 1, 2, \cdots, n$, and $g_{n+1}(x) = g_1(x)$. He proves two theorems concerning the Mellin transforms $R_p(s)$ and $L_q(s)$ of $r_p(x)$ and $l_q(x)$; for example: under certain conditions, we get $\prod_q L_q(s) = \prod_p R_p(s)$. *W. Saxer* (Zurich).

Thomsen, D. L., Jr. Extensions of the Laplace method. Proc. Amer. Math. Soc. **5**, 526–532 (1954).

Generalizing results of Fulks [same Proc. **2**, 613–622 (1951); these Rev. **13**, 233], the author considers the integral

$$I = \int_0^a \exp[-h\Phi(t) + k\Psi(t)] dt$$

as $h, k \to +\infty$ and $a \to 0+$. Under certain conditions concerning the functions $\Phi(t)$ and $\Psi(t)$, we have the relations

$$I \sim I_{a,h,k} \quad \text{or} \quad I \sim I'_{a,h,k},$$

where

$$I_{a,h,k} = \int_0^a \exp[-\tfrac{1}{2}h\Phi''(0)t^2 + k\Psi'(0)t] dt,$$

$$I'_{a,h,k} = \exp[-h\Phi(\tau) + k\Psi(\tau)] \int_0^a \exp[-\tfrac{1}{2}h\Phi''(0)(t-\tau)^2] dt$$

where $\tau = (k/h)\Psi'(0)/[\Phi''(0) + O(k^2/h^2)]$. We write $A \sim B$ when $(A-B)/B \to 0$. *W. Saxer* (Zurich).

Mitra, S. C., and Dinesh Chandra. On certain functions which are self-reciprocal under a new transform. Bull. Calcutta Math. Soc. **46**, 15–24 (1954).

The transform in question is that investigated by Bhatnagar [same Bull. **45**, 109–112 (1953); these Rev. **15**, 790; and earlier papers]. The present authors give a number of examples of self-reciprocal functions, give a formal derivation of Bhatnagar's characterization of self-reciprocal functions, state transformations which carry self-reciprocal functions into similar functions, etc. *A. Erdélyi.*

Kumar, Ram. A theorem on operational calculus. Bull. Calcutta Math. Soc. **46**, 37–40 (1954).

If $\phi(t)$ satisfies the relation

$$\phi(t) = t^{(\lambda/\mu)-1} \phi(t^{-1/\mu}) \exp[(b+c)t - bt^{-1/\mu}],$$

then the operational image of ϕ satisfies a certain integral equation. For $\mu = 1$, the kernel of the integral equation is a Bessel function. *A. Erdélyi* (Pasadena, Calif.).

Polynomials, Polynomial Approximations

Parodi, Maurice. Condition suffisante pour que tous les zéros finis de la dérivée du rapport de deux polynomes d'Hurwitz soient à partie réelle négative; applications aux matrices *H*. C. R. Acad. Sci. Paris **239**, 147–149 (1954).

Given two Hurwitz polynomials $f(z)$ and $g(z)$ of degrees n and m $(m<n)$ (viz., real polynomials with zeros only in the half-plane $\mathrm{Re}(z)<0$). If x_1, x_2, R_1, R_2 can be found so that $0<R_1<x_1$, $0<R_2<x_2$, $(x_1+R_1)(x_2+R_2)^{-1}<n/m$ and so that the circle C_1 with center $(-x_1, 0)$ and radius R_1 and the circle C_2 with center $(-x_2, 0)$ and radius R_2 enclose respectively all the zeros of $f(z)$ and $g(z)$, then also $F(z)=g^2(d/dz)(f/g)$ is a Hurwitz polynomial. The author proves this result immediately from a theorem due to Walsh [see M. Marden, Geometry of the zeros . . . , Math. Surveys, no. 3, Amer. Math. Soc., New York, 1949, p. 70, Theorem (20, 1); these Rev. 11, 101] and then applies it to the characteristic polynomials of a pair of square matrices.

M. Marden (Milwaukee, Wis.).

Butzer, P. L. On the extensions of Bernstein polynomials to the infinite interval. Proc. Amer. Math. Soc. **5**, 547–553 (1954).

The polynomials

$$P_n f(x)=e^{-nx}\sum_{\nu=0}^{\infty} f(\nu/n)(nx)^\nu/\nu!$$

are sometimes taken as the analogue for the interval $(0, \infty)$ of the Bernstein approximating polynomials. The theory was known for a continuous f; here the author discusses a modification for integrable f. His principal theorems are that if $f \in L(0, \infty)$ then

$$(*)\qquad ne^{-nx}\sum_{\nu=0}^{\infty}\left\{\int_{\nu/n}^{(\nu+1)/n} f(s)ds\right\}(nx)^\nu/\nu!\to f(x)$$

almost everywhere; if $f \in L$ and L^p, $p>1$, the convergence in (*) is dominated in L^p, and hence (*) also holds in the L^p sense. Some other polynomials for infinite intervals are discussed briefly.

R. P. Boas, Jr. (Evanston, Ill.).

Ossicini, Alessandro. Polinomi associati alle funzioni ultrasferiche di seconda specie. Boll. Un. Mat. Ital. (3) **9**, 170–177 (1954).

We denote by $P_n^{(\lambda)}(x)$ the ultraspherical polynomials and by $Q_n^{(\lambda)}(x)$ the associated functions. The relations

$$Q_n^{(\lambda)}(x)=\frac{\Gamma(\frac{1}{2})\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda)}\left[P_n^{(\lambda)}(x)\int_x^{\infty}(x^2-1)^{-\lambda-1}dx - (x^2-1)^{-\lambda+1}W_{n-1}^{(\lambda)}(x)\right]$$

define certain polynomials $W_{n-1}^{(\lambda)}(x)$ of degree $n-1$. Various properties of these polynomials are investigated, in particular their relation with $P_n^{(\lambda)}$ and $Q_n^{(\lambda)}$.

G. Szegő.

Singh, Vikramaditya. Appell set of polynomials. Proc. Nat. Inst. Sci. India **20**, 341–347 (1954).

Appell polynomial sets $\{P_n(x)\}$ (which can be defined by $A(t)e^{xt}=\sum_0^\infty P_n(x)t^n$) are characterized by certain integrals involving hypergeometric polynomials:

$$P_n(x)=\frac{x^n}{n!}\int_0^{\infty} {}_{r+1}F_r\left[\begin{matrix}-n, a_1, \cdots, a_r\\ b_1, \cdots, b_r\end{matrix}; -\frac{t}{x}\right]d\beta(t);$$

and the corresponding A-function is then given by

$$A(t)=\int_0^{\infty} {}_rF_r\left[\begin{matrix}a_1, \cdots, a_r\\ b_1, \cdots, b_r\end{matrix}; ut\right]d\beta(u).$$

Appell polynomials are also related to some well-known functions (e.g., Weber's cylindrical functions), and to Angelescu polynomials.

I. M. Sheffer.

Geronimus, Ya. L. On some asymptotic relations in the theory of orthogonal polynomials. Doklady Akad. Nauk SSSR (N.S.) **96**, 1097–1100 (1954). (Russian)

Let $P_n(z)$ be polynomials orthogonal on the unit circle with weight $d\sigma(\theta)$. The author establishes some limit relations involving subsequences of $\{P_n(z)\}$ and of the sequence of associated polynomials of the second kind. He deduces some connections between the $P_n(z)$ and properties of $\sigma(\theta)$, for example that $|P_{n_j}(z)|\leqq M$ for some subsequence $\{n_j\}$ implies $\sigma(\theta_2)-\sigma(\theta_1)\geqq(\theta_2-\theta_1)/M^2$.

R. P. Boas, Jr.

Clenshaw, C. W. Polynomial approximations to elementary functions. Math. Tables and Other Aids to Computation **8**, 143–147 (1954).

Gel'fond, A. On polynomials deviating least from zero along with their derivatives. Doklady Akad. Nauk SSSR (N.S.) **96**, 689–691 (1954). (Russian)

The author considers the problem of determining

$$\sigma_{n,m}=\min_{p(x)}\max_{0\leqq s\leqq m}\max_{-1\leqq x\leqq 1}|p^{(s)}(x)|(n-s)!/n!,$$

where $p(x)$ runs through real polynomials of degree not exceeding n, with leading coefficient 1. He shows that

$$2^{-n+m+1}\leqq\sigma_{n,m}\leqq 2^{-n+m+1}\exp\{O(m^2/n)\},$$

so that the problem is asymptotically solved when $m=o(n^{\frac{1}{2}})$. A related problem is discussed.

R. P. Boas, Jr.

Special Functions

Van Nostrand, Robert G. The orthogonality of the hyperboloid functions. J. Math. Physics **33**, 276–282 (1954).

The functions of the title are the associated Legendre functions $P^m_{ip-\frac{1}{2}}(\eta)$ of integral degree and real argument greater than unity. A proof of their orthogonality on the interval $(1, \infty)$ with weight function 1 is given together with an expansion of the reciprocal of the distance between two points in terms of these functions.

N. D. Kasarinoff (Lafayette, Ind.).

Toscano, Letterio. Carattere ipergeometrico dei polinomi associati a quelli di Hermite. Boll. Un. Mat. Ital. (3) **9**, 146–150 (1954).

Denoting by $H_n(x)$ the Hermite polynomials and by $h_n(x)$ the second solution of Hermite's differential equation, the relation

$$h_n(x)=h_0(x)H_n(x)-e^{x^2/2}G_{n-1}(x)$$

defines certain polynomials $G_{n-1}(x)$ which have been investigated previously by the author. In the present note these polynomials are represented in terms of confluent hypergeometric functions of two variables with a suitable numerical specification of one variable.

G. Szegő.

Rushton, S. On the confluent hypergeometric function $M(\alpha, \gamma, x)$. Sankhyā 13, 369–376 (1954).

Rushton, S., and Lang, E. D. Tables of the confluent hypergeometric function. Sankhyā 13, 377–411 (1954).

In the first of these papers the author describes the most important properties of the function

$$M(\alpha, \gamma, x) = \sum_{j=0}^{\infty} \frac{\Gamma(\gamma)\Gamma(\alpha+j)x^j}{\Gamma(\alpha)\Gamma(\gamma+j)j!},$$

its application to facilitate certain sequential tests of composite hypotheses, and the construction of numerical tables.

The second paper contains numerical tables of $M(\alpha, \gamma, x)$ to seven significant figures for

$\gamma = .5(.5)3.5, 4.5, \quad x = .02(.02).1(.1)1(1)10(10)50, 100, 200,$

and an extensive range of integer and half-integer values of α (this range being different for different values of γ and x). Tables of this function for $\gamma = 3, 4$ have been given by P. Nath [Sankhyā 11, 153–166 (1951); these Rev. 13, 494].

A. Erdélyi (Pasadena, Ill.).

Toscano, L. Equation différentielle de la fonction hypergéométrique confluente ψ_1 considérée comme fonction d'une seule variable. Bull. Soc. Roy. Sci. Liège 23, 164–170 (1954).

The function $\Psi_1(x, y)$ [Erdélyi et al., Higher transcendental functions, vol. 1, McGraw-Hill, New York, 1953, p. 225 (23); these Rev. 15, 419] satisfies a system of partial differential equations of the second order [ibid., p. 235 (26)]. As a function of x alone, the function satisfies an ordinary differential equation of the fourth order. The author obtains this differential equation, and also the corresponding equation for Ψ_1 as a function of y. As an application, he derives the ordinary differential equation of the fourth order satisfied by a product of Hermite polynomials.

A. Erdélyi (Pasadena, Calif.).

Abramowitz, Milton, and Antosiewicz, H. A. Coulomb wave functions in the transition region. Physical Rev. (2) 96, 75–77 (1954).

The authors represent a fundamental system of solutions of $d^2y/d\rho^2 + (1 - 2\eta/\rho)y = 0$ in the form

$$y_j(x, \mu) = \phi(x, \mu)u_j(x) + \psi(x, \mu)u_j'(x), \quad j = 1, 2,$$

where $x = (2\eta - \rho)/(2\eta)^{1/3}$, $\mu = (2\eta)^{2/3}$, and $u_1(x)$, $u_2(x)$ are linearly independent solutions of $u'' - xu = 0$. The functions ϕ and ψ turn out to be power series in x, and recurrence relations are available for the coefficients. The results are extended to the differential equation

$$d^2y/d\rho^2 + [1 - 2\eta\rho^{-1} - L(L+1)\rho^{-2}]y = 0.$$

The resulting expansions converge when $|\rho - \rho_1| < \rho_1$, where $\rho_1 = \eta + [\eta^2 + L(L+1)]^{1/2}$.

A. Erdélyi.

Abramowitz, Milton, and Rabinowitz, Philip. Evaluation of Coulomb wave functions along the transition line. Physical Rev. (2) 96, 77–79 (1954).

$F_0(\rho, \eta)$ and $G_0(\rho, \eta)$ are solutions of

$$d^2y/d\rho^2 + (1 - 2\eta/\rho)y = 0.$$

The authors apply the method of steepest descents to an integral representation of $F_0 - iG_0$ to obtain asymptotic expansions for F_0, G_0, F_0', G_0' when $\rho = 2\eta$ and $\eta \to \infty$. They give 7D tables of the quantities in question for

$$\rho = 2\eta = 0(.5)20(2)50,$$

and discuss the numerical computation of F_0 and G_0 for ρ near 2η.

A. Erdélyi (Pasadena, Calif.).

Bell, Dorothy G. Group theory and crystal lattices. Rev. Modern Physics 26, 311–320 (1954).

The paper gives first a short but self-contained presentation of the implications of crystal symmetries for the determination of one-electron wave functions. The angular parts of the wave functions around a nucleus are to be expressed in terms of the "lattice harmonics" for the rotation group leaving the lattice and the wave vector invariant. Tables of lattice harmonics for an angular momentum up to six are given for cubic and hexagonal close-packed lattices. The symmetry groups for wave vectors in the first Brillouin zone are also listed for the same lattices. *L. Van Hove.*

Harmonic Functions, Potential Theory

Tideman, M. Elementary proof of a uniqueness theorem for positive harmonic functions. Nordisk Mat. Tidskr. 2, 95–96 (1954).

Proof of the theorem: If $u(x, y)$ is a positive harmonic function for $y > 0$, and if $u(x, 0) = 0$, then $u(x, y) = cy$ for some constant $c > 0$.

Demidovič, B. P. A simple proof of the mean-value theorem for harmonic functions. Uspehi Matem. Nauk (N.S.) 9, no. 3(61), 213–214 (1954). (Russian)

Huckemann, Friedrich. On the "one-circle" problem for harmonic functions. J. London Math. Soc. 29, 491–497 (1954).

Suppose $u(x)$ is continuous in $-1 < x < 1$, $U(x) = \int_0^x u(t)dt$, and $\lim U(x)$ exists as $x \to 1$ and as $x \to -1$. If for every x there is an h, $-1 < x - h < x + h < 1$, such that

$$(*) \qquad 2hu(x) = U(x+h) - U(x-h),$$

then the author proves that $u(x) = Ax + B$. If (*) is replaced by $2u(x) = u(x+h) + u(x-h)$, the conclusion need not hold, even for bounded u. The title of the paper alludes to the two-dimensional analogue of this problem; reference is made to a theorem of Kellogg [Trans. Amer. Math. Soc. 36, 227–242 (1934), p. 233] but no further positive results are obtained.

W. Rudin (Rochester, N. Y.).

Moretti, Fiorenzia. Alcune diseguaglianze relative alle funzioni armoniche nel complementare di un dominio limitato ed infinitesime all'infinito. Boll. Un. Mat. Ital. (3) 9, 190–195 (1954).

Let D be an infinite domain with a boundary FD which is composed of a finite number of mutually exclusive smooth surfaces. Let ψ and χ be harmonic in D, zero at infinity, $\chi > 0$, $\partial\chi/\partial\nu > 0$ on FD (ν is directed to the exterior of D). Then

$$\int_{FD} \psi^2 d\sigma \leq H \int_{FD} (\partial\psi/\partial\nu)^2 d\sigma,$$

where $H = 4(\max \chi)^2(\min \partial\chi/\partial\nu)^{-2}$, max and min on FD. Also

$$\int_D |\operatorname{grad} \psi|^2 d\tau \leq H^1 \int_{FD} (\partial\psi/\partial\nu)^2 d\sigma.$$

The proofs are based on Green's formula. *G. Szegő.*

Cocchi, Giovanni. Campi potenziali attorno a schiere di cerchi. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 16, 42–47 (1954).

To approximate a potential function for which the family of circles of radius r and centers $z_k = \pm k\pi i$, $k = 0, 1, 2, \cdots$, are equipotential lines the author suggests that for small r/π a system of sources of strength m at z_k with complex potential $f(z) = \frac{1}{2}m \log \frac{1}{2}(1-\cosh z)$ may be used. For larger values of $r/\pi < 1$ he proposes to use sources of strength $\frac{1}{2}m$ at $z = \pm a + z_k$ with complex potential

$$f(z) = \tfrac{1}{2}m \log \tfrac{1}{2}(\cosh a - \cosh z).$$

If $2\cosh a = \cosh r + \cos r$, then all of the points $z = \pm r$ and $z = \pm ir$ are on the same closed equipotential curve l. For $r \leqq 2\pi/3$ the distance from the origin to any point of l is between r and $0.93r$. It is also shown that $Z^{-2} = b^{-2} - A \operatorname{sech} z$ maps the exterior of l in the strip $|y| \leqq \pi$ onto $|Z| \geqq 1$, where $A = 2\cosh r \cos r/(\cosh r - \cos r)$ and

$$b^{2e} = (\cosh r + \cos r)/(\cosh r - \cos r)$$

with $e = +1(-1)$ for $r < (>)\pi/2$. *J. H. Giese.*

Differential Equations

Nemyckiĭ, V. V. Some problems of the qualitative theory of differential equations. (Survey of contemporary literature.) Uspehi Matem. Nauk (N.S.) 9, no. 3(61), 39–56 (1954). (Russian)

This paper provides an excellent bird's eye view of the work done on the subject under discussion during the last five years. About 40% of the papers under review (totalling 58) refer to work done outside the USSR. The emphasis is still on the whole on Soviet research, which is rather natural under the circumstances, and indeed enhances the value for the non-Soviet reader. The various and well known topics of systems of the second order are discussed. About half of the paper is devoted, however, to a description of contributions to systems of higher order. *S. Lefschetz.*

Muller, G. M. On the indefinite integrals of functions satisfying homogeneous linear differential equations. Proc. Amer. Math. Soc. 5, 716–719 (1954).

Consider the differential operator $L = \sum_0^n p_i(z)D^{n-i}$ and a function $f(z)$, where $p_i(z)$ and $f(z)$ are analytic in some region of the complex plane in which $p_0(z) \neq 0$. An n-dimensional vector $\mathbf{h}(z)$ is said to be an integrating vector for the operator L with respect to $f(z)$ if for every solution $u(z)$ of $L[u] = 0$, $u(z)f(z) = d\{\sum_0^n h_i(z)u^{(i-1)}(z)\}/dz$. The author proves and discusses the following theorem: $\mathbf{h}(z)$ is an integrating vector for L with respect to $f(z)$ if and only if $h_i(z) = -\sum_0^{n-i}(-1)^j(p_{n-i-j}g)^{(j)}$, and v satisfies $L^*[v] = f(z)$, where L^* is the Lagrangian adjoint of L.

N. D. Kazarinoff (Lafayette, Ind.).

Latyševa, K. Ya. Concerning a result of N. S. Košlyakov. Izvestiya Akad. Nauk SSSR. Ser. Mat. 18, 207–212 (1954). (Russian)

The author discusses the existence of solutions of the form $y = x^\alpha P(x)$, α real, P polynomial, of equations of the type $\sum(p_i + sx)x^{n-i}y^{(n-i)} = 0$. In particular, she finds new cases where the equation of Jacobi admits such a type of solutions. *J. L. Massera* (Montevideo).

Biryuk, G. I. On an existence theorem for almost periodic solutions of certain systems of nonlinear differential equations with a small parameter. Doklady Akad. Nauk SSSR (N.S.) 96, 5–7 (1954). (Russian)

The author considers the system $dx/dt = Ax + \epsilon F(t, x)$, where ϵ is a small parameter,

$$|F(t, x') - F(t, x'')| \leqq L|x' - x''|,$$

and all the characteristic roots of A have positive real part. It is shown that if $F(t, x)$ is uniformly almost periodic with respect to t, then there is a unique almost-periodic solution to the nonlinear system. *R. Bellman.*

Otrokov, N. F. On the number of limit cycles of a differential equation in the neighborhood of a singular point. Mat. Sbornik N.S. 34(76), 127–144 (1954). (Russian)

The author considers the problem of finding the number of limit cycles in the neighborhood of a singular point of an equation $dy/dx = Q_N(x, y)/P_N(x, y)$, where P_N, Q_N are polynomials of degree N. The number of essential coefficients of P, Q (after certain canonical reductions) is

$$m = (N-1)(N+4)+1$$

and a one-one correspondence is thus established between equations and points in E_m. A point A_0 in E_m is called k-cyclic if: (i) there are numbers $\epsilon_0 > 0$, $\delta_0 > 0$ such that the equation corresponding to any point in the ϵ_0-neighborhood of A_0 does not have more that k limit cycles in the δ_0-neighborhood of the singular point $x = y = 0$; (ii) for any ϵ, δ, $0 < \epsilon < \epsilon_0$, $0 < \delta < \delta_0$, there are equations corresponding to points of the ϵ-neighborhood of A_0 which have k limit cycles in the δ-neighborhood of $x = y = 0$. The main result of the paper is that, for $N \geqq 6$, k-cyclic points exist with k at least equal to $\frac{1}{2}(N^2+5N-14)$, N even, or $\frac{1}{2}(N^2+5N-26)$, N odd. *J. L. Massera* (Montevideo).

Krasovskiĭ, N. N. On the behavior in the large of the integral curves of a system of two differential equations. Akad. Nauk SSSR. Prikl. Mat. Meh. 18, 149–154 (1954). (Russian)

Let $\dot{x} = X(x, y)$, $\dot{y} = Y(x, y)$, where X, Y have continuous first partials for any (x, y) (certain weaker assumptions are also considered), $X(0, 0) = Y(0, 0) = 0$; let J be the Jacobian matrix of X, Y at any point (x, y), $\lambda_i(x, y)$, $i = 1, 2$, its characteristic roots, $m(r) = \inf (X^2 + Y^2)^{1/2}$, $M(r) = \sup (X^2 + Y^2)^{1/2}$ for $x^2 + y^2 = r^2$. The author proves: (i) if $\operatorname{Re}\lambda_i < 0$ for $x^2 + y^2 \leqq R_1^2$ and $\int_R^{R_1} m(r)dr > 2\pi R M(R)$, then the circle $x^2 + y^2 \leqq R^2$ belongs to the attractive domain of $(0, 0)$; (ii) if $\operatorname{Re}\lambda_i < 0$ in the whole plane and $\int_0^\infty m(r)dr = \infty$, the origin is stable in the large; (iii) if the origin is the only singular point, $\operatorname{Re}\lambda_i(0, 0) > 0$, $\operatorname{Re}\lambda_i(x, y) < 0$ for $x^2 + y^2 = R^2$ and $\int_0^\infty m(r)dr = \infty$, there exists at least one stable limit cycle. *J. L. Massera* (Montevideo).

Łojasiewicz, S. Sur l'allure asymptotique des intégrales du système d'équations différentielles au voisinage de point singulier. Ann. Polon. Math. 1, 34–72 (1954).

The author considers the "asymptotic equivalence", as $t \to \infty$, of a linear system of real differential equations (1) $y' = Ay$ and of a non-linear conservative system (2) $x' = Ax + f(x)x$, where A is a constant matrix, the eigenvalues of which have negative real parts, and x, y, $f = f(x)$ are (column) vectors. "Asymptotic equivalence" of the two conservative systems means, for example, the existence of a C^1 topological mapping $y = \varphi(x)$, where $|x| < \epsilon$, with non-vanishing Jacobian which sets up a one-to-one correspondence of solutions of (1) and (2) in such a way that corre-

sponding solutions "coincide asymptotically". Solutions $y=y_0(t)$ and $x=x_0(t)$ are said to coincide asymptotically if for every point $(t_1, y_0(t_1))$ [or $(t_2, x_0(t_1))$] and every $\epsilon>0$, there exists a pair of numbers $\eta>0$, t_2 [or t_1] such that $E_2(t_2, \eta)\subset E_1(t_1, \eta)$ [or $E_1(t_1,\eta)\subset E_2(t_2, \eta)$], where $E_1(t_1, \eta)$ is the set of points (t, y) which are on solution paths $y=y(t)$ of (1) where $t\geq t_1$ and $|y(t_1)-y_0(t_1)|<\eta$.

The linear changes of variables $y=e^{At}z$ and $x=e^{At}w$ respectively transform (1) and (2) into (3) $z'=0$ and (4) $w'=G(t, w)$, where $G=e^{-At}f(e^{At}w)$. Although (4) is not conservative, (3) and (4) are essentially the case $A=0$ of (1) and (2). The main lemma establishes the "asymptotic equivalence" of (3) and (4) under certain conditions, which are too long to state here. They involve a condition of the type $|G(t, w_1)-G(t, w_2)|\leq\omega(t, |w_1-w_2|)$ occurring in Kamke's general uniqueness theorem. The result contains the theorem of Wintner [Amer. J. Math. 68, 125–132 (1946); these Rev. 7, 297] where it is assumed that $|G(t, w_1)-G(t, w_2)|\leq g(t)|w_1-w_2|$ and $\int^\infty g(t)dt<\infty$. (The conditions of the main lemma are also satisfied, for example, if $|G(t, w_1)-G(t, w_2)|\leq g(t)\psi(|w_1-w_2|)$, where $\int^\infty g(t)dt<\infty$ and $\int^\infty dr/\psi(r)=\infty$ [for a related result, cf. Wintner, ibid. 68, 451–454 (1946); these Rev. 8, 71].) The author's proof of the main lemma involves a change of the independent variable $t\to\tau$ in such a way that $t=\infty$ becomes $\tau=0$ and that the resulting system does not have a singularity at $\tau=0$ [for a similar procedure, cf., e.g., Lonn, Math. Z. 44, 507–530 (1938), pp. 526–527]. *P. Hartman.*

Evans, Robert L. Errors in asymptotic solutions of linear ordinary differential equations. Quart. Appl. Math. 12, 295–300 (1954).

Let $y(x)$ be a solution of a homogeneous, linear second-order differential equation with polynomial coefficients, and suppose $y(x)\sim\sum_0^\infty c_n x^{\rho-n}$ in a particular sector of the x-plane. The author shows how the differential equation satisfied by the function $v(x)=y(x)-\sum_0^{N-1} c_n x^{\rho-n}$ may be determined. This equation is linear and of first order so that $v(x)$ may be obtained as the integral of a known function. Thus, an upper bound for the error $v(x)$ made in approximating $y(x)$ by the first N terms of its asymptotic series is easily estimated. This upper bound generally turns out to be a sum of multiples of (and containing fractions) more than one of the neglected terms in the asymptotic series for $y(x)$. Moreover, the number of terms of this series giving the best approximation to $y(x)$ may be found. A specific equation having hypergeometric and Bessel subcases is treated. *N. D. Kazarinoff* (Lafayette, Ind.).

Šimanov, S. N. On the theory of oscillations of quasilinear systems. Akad. Nauk SSSR. Prikl. Mat. Meh. 18, 155–162 (1954). (Russian)

I. G. Malkin [same journal 14, 353–370 (1950); these Rev. 12, 101] proved that the system $\dot{x}=Ax+\mu f(t, x)$, x being an n-vector, A a constant matrix, f continuous and periodic in t and having continuous first-order partials which satisfy a Lipschitz condition, has periodic solutions which tend to certain periodic solutions of the linear system $\dot{x}=Ax$ when $\mu\to 0$. In the present paper it is shown that the Lipschitz condition on the partials of f may be omitted.

J. L. Massera (Montevideo).

Minorsky, N. La méthode stroboscopique et ses applications. Bull. Soc. Franç. Méc. 4, no. 13, 15–26 (1954).

This is a review paper with numerous references. The "stroboscopic method" is devised for the study of solutions of the system $x'=X(x, y, t)$, $y'=Y(x, y, t)$ in the neighborhood of a harmonic solution. The transformation $y=x'$, $\rho=x^2+y^2$, $\psi=\arctan(y/x)$ reduces the equation $x''+x=0$ to the system $\rho'=\psi'+1=0$, whence it is assumed that in general the system can be transformed into $\rho'=\epsilon f(\rho, \psi, t)$, $\psi'=1+\epsilon g(\rho, \psi, t)$, with ϵ a small parameter. One thereafter employs the method of variation of parameters, obtaining $\rho(t)=\rho_0+\epsilon\rho_1(t)+\cdots$, $\psi(t)=\psi_0(t)+\epsilon\psi_1(t)+\cdots$, with $\rho_0=$constant, $\psi_0(t)=\varphi_0-t$. Let

$$\rho_1(2\pi)=K(\rho_0, \varphi_0), \quad \psi_1(2\pi)=L(\rho_0, \varphi_0).$$

Then the periodic solutions of the original system are associated with the singular points of the system $d\rho/d\tau=K(\rho, \varphi)$, $d\psi/d\tau=L(\rho, \varphi)$. The method is discussed in general, and then applied to several equations, including the equations of Mathieu, and of van der Pol, and to certain differential-difference equations. *A. S. Householder.*

Gillies, A. W. On the transformations of singularities and limit cycles of the variational equations of van der Pol. Quart. J. Mech. Appl. Math. 7, 152–167 (1954).

The investigation of the periodic solutions of van der Pol's equation with a simple harmonic forcing term leads to the equations

$$(*) \qquad \dot{b}=b(1-b^2)-F\cos\phi, \quad b\dot{\phi}=-bx+F\sin\phi,$$

where b and ϕ are essentially the amplitude and phase of the oscillation. The present paper is concerned with the sequence of possible phase portraits of the solutions of (*) in the (b, ϕ) plane (b, ϕ are polar coordinates) as x varies with F held constant. The author indicates that for a range $\frac{1}{3}<F<F_p$ a sequence of transitions occurs as x decreases in which the limit cycle (which always exists for large x) disappears through a second-order singularity which forms on the cycle and then splits into two first-order singularities, one of which is a col or saddle point. For somewhat smaller x a new limit cycle forms from the col and then shrinks and coalesces with an unstable focus. In this range of F and for some $F\leq\frac{1}{3}$ the transitions of the phase portraits which the author proposes differ from the sequence of transitions proposed by Cartwright. From a study of the isoclines of (*) for certain critical values of x and F the author shows that his proposed sequence is considerably more plausible than Cartwright's. *C. E. Langenhop.*

Graffi, Dario. Sulle oscillazioni forzate nei sistemi non lineari a due gradi di libertà. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 16, 176–180 (1954).

Consider the system

$$L_1\ddot{x}_1+M\ddot{x}_2+g_1(\dot{x}_1)+K_1x_1=e_1\cos(\omega t+\gamma_1),$$
$$M\ddot{x}_1+L_2\ddot{x}_2+g_2(\dot{x}_2)+K_2x_2=e_2\cos(\omega t+\gamma_2).$$

By means of very simple calculations, the author finds bounds for the mean square value and the maximum value of $y_i=\dot{x}_i$ corresponding to a periodic solution. Certain consequences for stability are also derived. *J. L. Massera.*

Nehari, Zeev. On the zeros of solutions of second-order linear differential equations. Amer. J. Math. 76, 689–697 (1954).

Non-oscillation criteria are given for

$$(A) \qquad w''(z)+p(z)w(z)=0,$$

where $p(x)$ is regular in a domain D. Theorem 1: D is the half-strip $x\geq 0$, $|y|<1$; $p(x+iy)=o(|x|)$ and $xp[x\pm i(1-\eta)]$

is $L(0, \infty)$ for small $\eta>0$; then (A) is non-oscillatory in every half-strip $x\geqq 0$, $|y|\leqq 1-\delta$ ($\delta>0$). Theorem 3: D is $|z|<1$, and $\int_0^{2\pi}|p(e^{i\theta})|d\theta<\infty$; then (A) is non-oscillatory in $|z|<1$. Some further results are also proved.

G. E. H. Reuter (Manchester).

Hong, Imsik. On exceptional values of a solution of a differential equation. Kōdai Math. Sem. Rep. **1954**, 63–64 (1954).

Consider the differential equation

$$P_0(x)y''+P_1(x)y'+P_2(x)y=P_3(x),$$

where $P_\nu(x)$, $\nu=0, 1, 2, 3$, are polynomials in x. According to a theorem of Gross [Math. Ann. **78**, 332–342 (1918)], the Borel exceptional values of any non-constant analytic solution of the above equation cannot cover a continuum of values. The author sharpens this result by proving that the exceptional values always form a set of capacity zero.

O. Lehto (Helsinki).

Loĭcyanskiĭ, L. G. Free and forced vibrations with resistance laws which are quadratic and intermediate between linear and quadratic. Akad. Nauk SSSR. Inženernyĭ Sbornik **18**, 139–148 (1954). (Russian)

The equation $\ddot{\xi}\pm\frac{1}{2}\dot{\xi}^2+\xi=0$ of a dissipative motion in dimensionless coordinates is linearized to $\ddot{\xi}+2\nu\dot{\xi}+\xi=0$ and discussed in standard manner.

S. Lefschetz.

Vinograd, R. È. On boundedness of solutions of regular systems of differential equations with small added terms. Uspehi Matem. Nauk (N.S.) **8**, no. 1(53), 115–120 (1953). (Russian)

The author considers the vector-matrix equation

$$x'=A(t)x+f(t, x),$$

where $f(t, 0)=0$, and $\|f(t, x_1)-f(t, x_2)\|\leqq g(t)\|x_1-x_2\|$ with $\int^\infty g(t)dt<\infty$. A number of cases where the boundedness of solutions of the linear system entails the same property of the nonlinear system are discussed, and an example, in the case of variable $A(t)$, where this does not hold.

R. Bellman (Santa Monica, Calif.).

Vinograd, R. È. Instability of characteristic exponents of regular systems. Doklady Akad. Nauk SSSR (N.S.) **91**, 999–1002 (1953). (Russian)

An example due to Perron [J. Reine Angew. Math. **143**, 25–50 (1913)] shows that there exist differential equations of the form $y'=A(t)y$ with the property that all solutions are bounded and matrices $B(t)$ satisfying the conditions $B(t)\to 0$, $\int^\infty\|B(t)\|dt<\infty$, such that not all solutions of $x'=(A(t)+B(t))x$ are bounded. In the paper reviewed above the author constructs another example of this instability phenomenon. In the present paper he constructs a stronger example which shows that $\lim_{t\to\infty}\log\|y\|/t$ can exist and be negative for all solutions of the unperturbed equation, and yet solutions of the perturbed equation can exist for which $\limsup_{t\to\infty}\log\|x\|/t$ is positive.

R. Bellman.

Veiga de Oliveira, F. Characteristic exponents. Application to stability. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) **2**, 201–288 (1952); **3**, 5–59 (1954). (Portuguese. French summary)

This is a very clearly written expository article. Vector-matrix notation is used.

J. L. Massera (Montevideo).

Lebedev, A. A. On the problem of stability of motion over a finite interval of time. Akad. Nauk SSSR. Prikl. Mat. Meh. **18**, 75–94 (1954). (Russian)

Lebedev, A. A. On stability of motion during a given interval of time. Akad. Nauk SSSR. Prikl. Mat. Meh. **18**, 139–148 (1954). (Russian)

The author develops certain ideas of G. V. Kamenkov [same journal **17**, 529–540 (1953); these Rev. **15**, 795]. Different types of stability over a finite interval of time such as stability under finite initial perturbations and under finite constantly acting perturbations are discussed in the first paper; in the second one the ideas of the so-called second method of Lyapunov are applied to this problem.

J. L. Massera (Montevideo).

Ermolaev, L. On uniform stability in the first approximation of denumerable almost linear and nonlinear systems of differential equations. Izvestiya Akad. Nauk Kazah. SSR **1952**, no. 116, Ser. Astr. Fiz. Mat. Meh. **1**(6), 88–105 (1952). (Russian. Kazak summary)

Ermolaev, L. Uniform stability of a denumerable almost linear system of differential equations whose linear part has triangular form. Izvestiya Akad. Nauk Kazah. SSR **1952**, no. 116, Ser. Astr. Fiz. Mat. Meh. **1**(6), 106–114 (1952). (Russian. Kazak summary)

Systems (1): $\dot{y}=\Omega(t, y)=P(t)y+f(t, y)$ are considered, where y is a vector with a denumerable number of real or complex components y_i, $\|y\|=\sup|y_i|$, Ω real or complex, continuous in t, $\|\Omega(t, y')-\Omega(t, y'')\|\leqq A(t)\|y'-y''\|$, $A(t)$ continuous, $\Omega(t, 0)=0$, $P=(p_{ik})$ a continuous matrix, $\sum_k|p_{ik}(t)|\leqq p(t)$, $p(t)$ continuous,

$$\|f(t, y')-f(t, y'')\|\leqq C(t)\|y'-y''\|,\quad \int_0^\infty C(t)dt<\infty.$$

Typical results are: (i) for the uniform stability of the solution $x=0$ of (1) it is necessary and sufficient that the linear approximation (2): $\dot{x}=Px$ be uniformly stable; (ii) if the solutions of (2) satisfy (3): $\|x(t)\|\leqq\|x(t_0)\|\cdot B(t_0)$, $\int_0^\infty B(t)C(t)dt<\infty$, the solution $x=0$ of (1) is stable; (iii) if a function $V(t, x)$ exists such that $V\geqq 0$, $V(t, 0)=0$, V has an infinitely small upper bound, $(\partial V/\partial t)+(\partial V/\partial x)\cdot Px\leqq 0$, the solution $x=0$ of (1) is uniformly stable; (iv) if the characteristic numbers of the solutions of (2) are non-negative and (3) is satisfied, the solution $x=0$ of (1) is stable; (v) if P is constant or periodic and the characteristic numbers of the solutions of (2) are $\geqq 0$, the solution $x=0$ of (1) is uniformly stable under constantly acting perturbations; (vi) if P is triangular ($p_{ik}=0$ for $s<k$),

$$p(t)\leqq n,\quad \left|\exp\int_{t_0}^t p_{ss}(\tau)d\tau\right|\leqq N,$$

$$\left|\exp\int_{t_0}^t p_{ss}(\tau)d\tau\right|\cdot\int_{t_0}^t\left|\exp-\int_{t_0}^\tau p_{ss}(\tau)d\tau\right|d\tau\leqq M,\quad mM\leqq\gamma<1,$$

the solution $x=0$ of (1) is uniformly stable. Special consequences of this last theorem are derived for the cases $P=$constant or $p_{ss}(t)$ periodic.

J. L. Massera.

Barbuti, Ugo. Su alcuni teoremi di stabilità. Ann. Scuola Norm. Super. Pisa (3) **8**, 81–91 (1954).

The author shows how the Liouville transformation, $y=R\cos v$, where $v=\int^t\sqrt{B(s)}\,ds$, $R=B^{-1/4}$, applied to the equation $y''+B(t)y=0$, permits one to obtain a number of results concerning stability and asymptotic behavior in a very simple and direct fashion.

R. Bellman.

Lyaščenko, N. Ya. On asymptotic stability of solutions of a system of differential equations. Doklady Akad. Nauk SSSR (N.S.) **96**, 237–239 (1954). (Russian)

The author considers a linear system $dx/dt=A(t)x$, where $A(t)$ is "close" in a sense made precise by the author, to a matrix $B(t)$ all of whose characteristic roots as functions of t are uniformly in the left half-plane. Under certain conditions which are rather complicated all the solutions of the original equation go to zero exponentially as $t\to\infty$.

R. Bellman (Santa Monica, Calif.).

Klotter, K., et Pinney, E. Un critère étendu de la stabilité des vibrations forcées dans les systèmes non linéaires. Rev. Gén. Electricité **63**, 559–562 (1954).

Translation of J. Appl. Mech. **20**, 9–12 (1953); these Rev. **14**, 647.

Bautin, N. N. A dynamic model of a watch movement without a characteristic period. Akad. Nauk SSSR. Inženernyĭ Sbornik **16**, 3–12 (1953). (Russian)

A standard simplified model of a clock with the pendulum replaced by a continuous band with stops. The period and amplitude of the oscillations are determined.

S. Lefschetz (Princeton, N. J.).

Coddington, E. A. The spectral representation of ordinary self-adjoint differential operators. Ann. of Math. (2) **60**, 192–211 (1954).

Let in $L=p_0(d/dx)^n+p_1(d/dx)^{n-1}+\cdots+p_n$ the coefficients p_k be complex-valued functions with $(n-k)$ continuous derivatives on an open interval (a, b) on the real line with $|p_0(x)|\neq 0$ on (a, b), and such that

$$L=L^+=(-1)^n(d/dx)^n(\bar{p}_0\cdot)+(-1)^{n-1}(d/dx)^{n-1}(\bar{p}_1\cdot)+\cdots+\bar{p}_n\cdot.$$

Here the case $-\infty=a$ or $b=\infty$ or both are not ruled out. The author gives explicit information concerning the spectral resolution associated with self-adjoint operators which arise from L. In this way, he extends the Weyl-Stone-Titchmarsh-Kodaira theory of eigenfunction expansion for the case of $n=2$ with real coefficients p_k. [The case of even order with real coefficients p_k was treated, as the author states, by Kodaira in Amer. J. Math. **72**, 502–544 (1950); these Rev. **12**, 103.]

Let T be the largest operator in $L_2(a, b)$ for which $Tu=Lu$ makes sense in $L_2(a, b)$, and let T_0 be its adjoint. Without making explicit use of the Green's function of L, it is proved that the set of all self-adjoint extensions of T_0 can be placed in a one-one correspondence with a certain class of self-adjoint boundary conditions of L. The following two special cases correspond respectively to the limit point case and the limit circle case in Weyl-Stone-Titchmarsh-Kodaira's theory: (i) $T_0=H$ is itself self-adjoint; (ii) one of the end points, say a, is finite and a self-adjoint extension H of T_0 can be determined only by the boundary condition at a. This second case can occur only when n is even, and $n=2\omega$ where (ω, ω) is the deficiency index of T_0. In each of these cases, it is proved that $E(\lambda)-E(\mu)$, where $H=\int\lambda dE(\lambda)$, can be regarded as an integral operator with a kernel given in terms of a linearly independent set of solutions of the equation $Lu=\lambda u$ and an essentially uniquely determined spectral matrix ρ. The existence of at least one such matrix is due to an earlier paper by B. M. Levitan [Doklady Akad. Nauk SSSR (N.S.) **73**, 651–654 (1950); these Rev. **12**, 502]. The author does not make direct use of the Green's function of L, and hence he dispenses with the restriction that n be even and p be real.

K. Yosida (Osaka).

Partial Differential Equations

***Petrovskiĭ, I. G. Lekcii ob uravneniyah s častnymi proizvodnymi. [Lectures on partial differential equations.]** 2d ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953. 360 pp. 7.20 rubles.

Although the general character of this edition is like the first [1950; these Rev. **13**, 241], there have been numerous small changes, two sections have been rewritten, and an appendix concerning the method of finite differences has been added. The author credits these major changes to O. A. Oleĭnik.

Bouligand, Georges. Cas limites d'équations

$$f(x, y, z, p, q; m)=0.$$

C. R. Acad. Sci. Paris **238**, 2209–2211 (1954).

Das Cauchysche Problem der partiellen Differentialgleichung $F(x, y, z, p, q)=0$ für die Anfangskurve $x(v)$, $y(v)$, $z(v)$ kann gemäss der Substitutionen $p=A(x, y, z, u)$, $q=B(x, y, z, u)$ und der Bedingung

$$\Omega=A_u\frac{\partial u}{\partial y}-B_u\frac{\partial u}{\partial x}+(BA_u-AB_u)\frac{\partial u}{\partial z}+A_v-B_z-AB_z=0$$

auf das Cauchysche Problem der partiellen Differentialgleichung $\Omega=0$ für die Kurve $x(v)$, $y(v)$, $z(v)$, $u(v)$ zurückgeführt werden, wenn $u(v)$ der Bedingung

$$A[x(v), y(v), z(v), u]x'(v)+B[x(v), y(v), z(v), u]y'(v)=z(v)$$

unterworfen wird. In diesem Zusammenhang nimmt Verfasser jetzt wiederum die Untersuchung des Falles

$$f(x, y, z, p, q, m)=0$$

auf, worin m einen beliebig kleinen Parameter bedeutet. Damit gelingt jetzt Verfasser die Bestätigung früherer damals indirekt bewiesener Ergebnisse [vgl. G. Bouligand, J. Math. Pures Appl. (9) **16**, 251–266 (1937), insbesondere S. 257–259] auf direktem Wege.

M. Pinl (Köln).

***Sauer, R. Remarques géométriques sur les équations aux dérivées partielles du second ordre quasilinéaires et homogènes.** Premier colloque sur les équations aux dérivées partielles, Louvain, 1953. pp. 119–126. Georges Thone, Liège; Masson & Cie, Paris, 1954.

Geometrical interpretations of known results involving the connection between the characteristic curves of the equations mentioned in the title and those of the equations obtained from them by the Legendre transformation.

D. Gilbarg (Stanford, Calif.).

Cimmino, Gianfranco. Sulle equazioni lineari alle derivate parziali di tipo ellittico. Rend. Sem. Mat. Fis. Milano **23** (1952), 183–203 (1953).

This is an expository account of a method for obtaining existence theorems. The main part of the article deals with Dirichlet's problem for Poisson's equation in a bounded domain D with a twice continuously differentiable boundary FD. The proof is essentially a specialisation of one given by the author in Rend. Sem. Mat. Univ. Padova **11**, 28–89 (1940) [these Rev. **8**, 270]. It employs the Hilbert space of pairs of functions f, ϕ defined in D and on FD, respectively,

with the distance defined by $\int\int_D f^2 dxdy + \int_{\Gamma_D} \phi^2 ds$. Accordingly, the author arrives at weak existence statements for a generalised problem. No attempt is made to pass from there to the classical existence theorems. Other problems are briefly touched and several existence theorems are indicated in which the method has been successfully applied.

W. Wasow (Rome).

Gergen, J. J., and Dressel, F. G. Mapping for elliptic equations. Trans. Amer. Math. Soc. 77, 151–178 (1954).

Le but de cet article est d'étendre le théorème de Riemann sur la représentation conforme aux transformations $X=u(x, y)$, $Y=v(x, y)$ qui satisfont à:

$$(1) \qquad \alpha u_x+\beta u_y=v_y, \quad \gamma u_x+\delta u_y=-v_x,$$

$\alpha, \beta, \gamma, \delta$ étant des fonctions de classe C^1 satisfaisant à $\alpha>0$, $4\alpha\delta-(\beta+\gamma)^2>0$. A cet effet on étudie les 2 cas particuliers

$$(1a) \qquad \beta+\gamma=0, \qquad \delta=\alpha,$$
$$(1b) \qquad \beta=\gamma, \quad \alpha\delta-\beta^2=1,$$

car toute transformation du type général (1) peut être considérée comme le produit de 2 transformations de types respectifs (1a) et (1b).

Les transformations du type (1b) définissent la représentation conforme sur le plane (u, v) de la surface d'élément linéaire $ds^2=\delta dx^2-2\beta dxdy+\alpha dy^2$; elles ont été étudiées par C. B. Morrey [mêmes Trans. 43, 126–166 (1938)]. Les transformations du type (1a) généralisent les représentations p-régulières des auteurs [Duke Math. J. 18, 185–210 (1951); 19, 435–444 (1952); ces Rev. 14, 262] qui correspondent au cas $\beta=\gamma=0$, $\alpha=\delta=p$. On considère ici la transformation inverse $x=U(x, y)$, $y=V(x, y)$ comme définissant une solution particulière, dans le cas $m=2$, du système:

$$(2) \qquad V_{y_i}=\sum_{k=1}^{m} p_{ik}U_{x^k}, \quad -V_{x^i}=\sum_{k=1}^{m} p_{ik}U_{y^k}$$

avec $p_{ik} \in C^1$, $p_{ik}=p_{ki}$ et la forme $p_{ik}x^ix^k$ étant définie positive.

La partie I est consacrée à l'étude du système (2); le problème aux limites est ramené à une équation intégrale, qui en donne la solution, et rattaché au problème de minimum de l'intégrale $\int\int_D p_{ik}\,\mathrm{grad}\,w^i \cdot \mathrm{grad}\,w^k dS$; puis on étudie la continuité des solutions. Les résultats obtenus, joints à ceux de Morrey, permettent de résoudre le cas (1a) [partie II]. La partie III apporte le résultat final: Soient D et T deux domaines simplement convexes limités par des courbes de Jordan D^* et T^*; les fonctions $\alpha, \beta, \gamma, \delta$ de classe C^1 étant données et satisfaisant à $\alpha\delta-\frac{1}{4}(\beta+\gamma)^2>\mu>0$ ($\mu=$cte), il existe une représentation continue et biunivoque de $\bar{D}$ sur $\bar{T}$, de classe C^1 dans D satisfaisant à (1) et faisant correspondre 3 couples de points frontières donnés. Le problème de l'unicité n'es pas abordé.

J. Lelong (Lille).

Martin, M. H. A generalization of the method of separation of variables. J. Rational Mech. Anal. 2, 315–327 (1953).

If the method of separation of variables is applied to Laplace's equation $u_{xx}+u_{yy}=0$, the problem of integration is reduced to the differential equation

$$(*) \qquad X''Y+XY''=0,$$

or to the integration of the ordinary differential equations:

$$X''-k^2X=0, \quad Y''+k^2Y''=0.$$

The relation (*) is a special case of the relation:

$$(**) \qquad x_1(x)y_1(y)-x_2(x)y_2(y).$$

Introducing the curves

$$M: \quad x_1=x_1(x), \quad x_2=x_2(x); \qquad N: \quad y_1=y_1(y), \quad y_2=y_2(y),$$

the author shows: the necessary and sufficient condition for the relation (**) to hold as identity is that $M \in S$, $N \in T$, where

$$S: \quad b_1x_1+b_2x_2=0, \qquad T: \quad a_1y_1+a_2y_2=0$$

are two straight lines perpendicular to each other at the origin, or that one of M, N be the origin itself, in which case the other can be taken arbitrarily. A generalization to (**) is the separable identity

$$\sum_{\alpha=1}^{n} x_\alpha(u_1, \cdots, u_r)y_\alpha(v_1, \cdots, v_s)\equiv 0,$$

with which are associated the manifolds:

$$M: \quad x_\alpha=x_\alpha(u_1, \cdots, u_r), \qquad N: \quad y_\alpha=y_\alpha(v_1, \cdots, v_s)\equiv 0.$$

The author frames the following questions. (I) What can be said about the manifolds M, N, if the functions $x_\alpha(u_1, \cdots, u_r)$, $y_\alpha(v_1, \cdots, v_s)$ are required to satisfy a separable identity? (II) Can the answer to (I) be used to obtain particular solutions of partial differential equations, as was the case in the simple example above? (III) Can the usual method of separation of variables, in which solutions of the form $u=X(x)Y(y)$ are sought for, be extended to the search for solutions with the form $u=u(X(x), Y(y))$? A pair of points x, y will be termed orthogonal if their coordinates x_α, y_α are bound together by the bilinear relation $\sum_{\alpha=1}^{n} x_\alpha y_\alpha=0$. Two point sets M, N are orthogonal if each pair of points $x \in M$, $y \in N$ is orthogonal. The answer to question (I) is contained in the following lemma: The necessary and sufficient conditions for two point sets M, N to be orthogonal is that $M \subset S_p$, $N \subset T_q$ where S_p, T_q are linear orthogonal subspaces of respective dimensions p, q, with $p+q=n$. The linear subspaces S_p, T_q defined by

$$S_p: \sum_{\alpha=1}^{n} b_{k\alpha}x_\alpha=0, \ k=1, \cdots, q; \qquad T_q: \sum_{\alpha=1}^{n} a_{i\alpha}y_\alpha=0, \ i=1, \cdots, p,$$

where the matrix $A=\|a_{i\alpha}\|$ has rank p and the matrix $B=\|b_{k\alpha}\|$ has rank q will be orthogonal if and only if

$$A_{i_1 \cdots i_p}/B_{k_1 \cdots k_q}=\text{const.},$$

where $A_{i_1 \cdots i_p}$ is the minor comprising the columns $i_1, \cdots, i_p$ of A and $B_{k_1 \cdots k_q}$ is the minor formed by the columns $k_1, \cdots, k_q$ of B with $i_1, \cdots, i_p, k_1, \cdots, k_q$ forming an even permutation of $1, \cdots, n$. As a partial answer to questions (II) and (III) the author shows: The only solution of Laplace's equation $u_{xx}+u_{yy}=0$ having the form $u=u(U)$, $U=X(x)+Y(y)$, in which $u(U)$, $X(x)$, $Y(y)$ are real functions of their arguments, are obtained when $X(x)$, $Y(y)$ are integrals of elliptic functions (or their degenerate cases) and $u(U)$ is one of the elementary functions in the table given in the text. Finally some complex solutions are discussed and further extensions.

M. Pinl (Cologne).

Browder, Felix E. The eigenfunction expansion theorem for the general self-adjoint singular elliptic partial differential operator. I. The analytical foundation. Proc. Nat. Acad. Sci. U. S. A. 40, 454–459 (1954).

Browder, Felix E. Eigenfunction expansions for singular elliptic operators. II. The Hilbert space argument; parabolic equations on open manifolds. Proc. Nat. Acad. Sci. U. S. A. 40, 459–463 (1954).

Although not referred to, von Neumann's theory of continuous sums [Ann. of Math. (2) 50, 401–485 (1949); these Rev. 10, 548] is the natural background to these two notes.

In the simplest terms, a continuous sum $S=S(\mu, \nu)$ is a Hilbert space characterized by a measure μ on the real axis Λ and a measurable dimension function ν taking the values $1, 2, \cdots$ and ∞. The elements F of S are all (equivalence classes of) vector-valued functions $F(\lambda)=(F_1(\lambda), \cdots)$ with $\nu(\lambda)$ measurable complex components, the scalar product being

$$(F, G)=\int_\Lambda \sum F_k(\lambda)\overline{G_k(\lambda)}d\mu(\lambda).$$

The spectral theorem of von Neumann can be stated as follows. Given a self-adjoint operator A on a separable Hilbert space H, there exists a continuous sum $S=S(\mu, \nu)$ and a unitary mapping V from H to S which diagonalizes A in the sense that $(VAV^{-1}F)(\lambda)=\lambda F(\lambda)$. By this property, μ and ν are determined up to equivalence. In the elementary case when H is finite-dimensional, μ is concentrated in the eigenvalues of A and the components F_k of $F=Vf$ can be computed by means of the formula $F_k(\lambda)=(f, s_k)$ where $s_k=s_k(\lambda)$ is a suitable eigenvector of A belonging to the eigenvalue λ. The inverse formula (the Fourier expansion) reads $f=\int\sum s_k(\lambda)F_k(\lambda)d\mu(\lambda)$. The main result of the two notes is that in a certain sense, these formulas are still true when H is the space of all square-integrable functions on an open subset D of real n-dimensional space with values in a finite-dimensional Hilbert space and A is an extension of an elliptic differential operator L with sufficiently smooth coefficients. In that case $F_k(\lambda)=\int_D(f(x), s_k(\lambda, x))dx$ when f vanishes outside a compact set. The function $s_k(\lambda, x)$ is measurable $d\mu\times dx$ and for almost all λ it is a smooth eigenfunction of L, $Ls_k(\lambda, x)=\lambda s_k(\lambda, x)$, but not necessarily square integrable. Similarly, if F vanishes outside a compact set and $f=V^{-1}F$, then $f(x)=\int\sum s_k(\lambda, x)F_k(\lambda)d\mu(\lambda)$ holds a.e. The proof, which uses an idea of Mautner [same Proc. 39, 49–53 (1953); these Rev. 14, 659], is based on the fact that if k is large enough, $(A+i)^{-k}$ is given by a Carleman kernel. (Essentially the same theorem was also proved by the reviewer in Tolfte Skand. Matematikerkongr., Lund, 1953, pp. 44–55 (1954). He also gave a different proof in an unpublished lecture in 1954 at the University of Maryland.) The set of eigenfunctions s_k determines the self-adjoint extension A. If A is bounded from below, the parabolic equation $du/dt+Au=0$, $u\in H$, $u(0)=g$, has the solution $u=V^{-1}e^{-\lambda t}Vg$. It is shown that u is a smooth function.

L. Gårding (Lund).

Grinberg, S. I. On the asymptotic behavior of the eigenvalues of the Laplace operator. Uspehi Matem. Nauk (N.S.) 8, no. 6(58), 97–102 (1953). (Russian)

The author considers $\Delta u+\lambda u=0$, where Δ is the Laplace operator in 3 dimensions, and the eigenvalues are $\{\lambda_j\}$ for the Dirichlet problem on a finite domain D bounded by a surface S having positive curvature everywhere. For this case the author sharpens the formula of Carleman for $J=\sum_1^n(\lambda_j^2+\lambda_j p)^{-1}$, where p is large and positive by showing that

$$J=V(4\pi p^4)^{-1}-S(16\pi p)^{-1}\ln p+O(p^{-1}),$$

where V depends on D and S is the area of the surface.

N. Levinson (Cambridge, Mass.).

Hunt, G. A. On positive Green's functions. Proc. Nat. Acad. Sci. U. S. A. 40, 816–818 (1954).

Dans un espace euclidien E^k, d'élements $x(x_1, \cdots, x_j, \cdots, x_k)$, pour un opérateur elliptique

$$\Delta u=\sum(\partial/\partial x_i)[a_{ij}(x)\partial u/\partial x_j],$$

où $a_{ij}=a_{ji}$, une fonction de Green (nulle à l'infini) positive est une fonction $K(x, y)$ telle que: $\lim_{x\to\infty}K(x, y)=0$, $\Delta\int_{E^k}K(x, y)f(y)dy=-f(x)$ pour toute fonction $f(x)$ régulière nulle en dehors d'un compact. Posons $|x|=(\sum x_j^2)^{1/2}$; l'auteur montre que: si (a) il existe une fonction positive $A(r)$ telle que $\int_1^\infty r^{1-k}A(r)dr<\infty$ et que $A(|x|)\sum a_{ij}\lambda_i\lambda_j\geqq\sum\lambda_j^2$ pour tout $x\in E^k$ et quels que soient les nombres λ; (b) les a_{ij} et les $\partial a_{ij}/\partial x_k$ satisfont à des conditions de Lipschitz d'ordre 1; alors Δ possède une fonction de Green positive.

Pour sa démonstration, l'auteur s'appuie sur un théorème (dont la démonstration n'est par publiée) selon lequel, dans les conditions indiquées, l'équation $\partial u/\partial t=\Delta u$ détermine dans E^k un processus de Markoff homogène presque-sûrement continu, de densité de probabilité de passage $p(t; x, y)$; $K(x, y)=\int_0^\infty p(t; x, y)dt$ est alors une fonction de Green positive pour Δ.

R. Fortet (Paris).

Peters, A. S., and Stoker, J. J. A uniqueness theorem and a new solution for Sommerfeld's and other diffraction problems. Comm. Pure Appl. Math. 7, 565–585 (1954).

This paper is concerned with the uniqueness of boundary-value problems for the reduced wave equation $\nabla^2f+f=0$ in two dimensions. Such problems arise in the theory of diffraction of scalar monochromatic waves; a primary wave is incident on certain obstacles, and it is desired to find the secondary wave which represents the effect of reflection and diffraction and also to show that the secondary wave is uniquely determined.

If the obstacles lie entirely at a finite distance, Sommerfeld's radiation condition at infinity together with the boundary conditions determines the solution uniquely.

If, however, the obstacles extend to infinity, the secondary waves may not satisfy the radiation condition; a simple instance is the reflection of a plane wave at the whole x axis, another the reflection and diffraction of a plane wave by a reflector along the positive half of the x-axis (the Sommerfeld problem).

The uniqueness theorem proved here applies to a domain D, not necessarily simply connected, whose boundary Γ extends to infinity; it is assumed that, outside a sufficiently large circle, the boundary consists of a single half-ray going off to infinity. It is assumed that f satisfies certain boundary and regularity conditions on Γ and that it can be expressed in the form $g+h$, where h satisfies the radiation condition and g represents the primary wave and any secondary waves which do not satisfy the radiation condition. Moreover, at infinity $g\sim g_1+g_2$ where g_1 is prescribed once for all but g_2 satisfies the radiation condition. Under these circumstances f is uniquely determined.

This theorem seems at first sight unsatisfactory to the authors, since the leading term g_1 of the asymptotic expression for g is not given a priori. Yet in some instances it is easy to guess how g_1 should be defined. For instance, in Sommerfeld's problem, it would be natural to take g_1 to be the discontinuous field of geometrical optics, to take $g_2=0$, and to take h to be a discontinuous wave function which satisfies the radiation condition. It is, in fact, shown that this decomposition $f=g_1+h$ leads to a direct and straightforward way of solving Sommerfeld's problem.

The ideas involved are extensions of earlier work by F. Rellich [Jber. Deutsch. Math. Verein. 53, 57–65 (1943); these Rev. 8, 204] and F. John [Comm. Pure Appl. Math. 3, 45–101 (1950); these Rev. 12, 214].

E. T. Copson.

Calderón, A. P. The multipole expansion of radiation fields. J. Rational Mech. Anal. 3, 523–537 (1954).

The author considers the solution of the vector equation

$$\operatorname{curl}\operatorname{curl}\mathbf{E}=c^2\mathbf{E}$$

where c is a real constant, in a connected domain $\mathfrak{D}$ whose complement $\mathfrak{R}$ is the union of a finite number of bounded regions. $\mathfrak{D}$ and $\mathfrak{R}$ have a common boundary Σ consisting of a finite number of disjoint smooth surfaces. $\mathbf{E}$ has no singularities in $\mathfrak{D}$ and its tangential component on Σ has to take prescribed values; moreover, $\mathbf{E}$ has to satisfy a radiation condition at infinity in $\mathfrak{D}$.

The final result is as follows: Let p_i be fixed points, one in each component of $\mathfrak{R}$. Let $\Gamma(p, q)$ be the gradient of $(2\pi)^{-1}|p-q|^{-1}\exp(ic|p-q|)$ with respect to q. Let $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ be constant linearly independent vectors. Consider the fields defined by

$$\mathbf{E}_{ij}(p)=\Gamma(p_i, q)\times\mathbf{a}_j$$

and all their derivatives with respect to the coordinates of p_i. Let x be any point on Σ. Let $\mathfrak{L}^2$ be the Banach space of all $\mathbf{e}(x)$ tangential to Σ such that

$$\|\mathbf{e}\|=\left[\int_\Sigma|\mathbf{e}(x)|^2dx\right]^{\frac{1}{2}}<\infty.$$

Then the space $\mathfrak{L}^2$ is spanned by the tangential components of the fields just defined.

It follows that we can normalize these fields and obtain a sequence $\mathbf{E}_n^*(p)$ such that, when x is on Σ and $\mathbf{n}_x$ is the unit vector at x normal to Σ, the sequence $\mathbf{E}_n^*(x)\times\mathbf{n}_x$ is a complete orthonormal system of tangential vectors on Σ. Then given any $\mathbf{e}$ in $\mathfrak{L}^2$, we have

$$\mathbf{e}(x)=\sum_1^\infty c_n\mathbf{E}_n^*(x)\times\mathbf{n}_x,\quad c_n=\int_\Sigma\mathbf{e}(x)\cdot\overline{\mathbf{E}}_n^*(x)\times\mathbf{n}_x dx,$$

where the series converges to $\mathbf{e}(x)$ in the mean of order 2. The series $\sum_1^\infty c_n\mathbf{E}_n^*(p)$ satisfies the boundary-value problem for $\mathbf{e}(x)$ belonging to $\mathfrak{L}^2$; for it converges to the field $\mathbf{E}(p)$ defined by

$$\int_\Sigma|\mathbf{E}(x+\lambda\mathbf{n}_x)\times\mathbf{n}_x-\mathbf{e}(x)|^2dx\to0\quad\text{as}\quad\lambda\to0$$

uniformly at a positive distance from Σ.

It is impossible to give in a brief space the enunciation of the approximation theorems and existence theorems proved in the course of the paper, still less to give any indication of the interesting use made of Banach-space technique in this fundamental problem of radiation theory. *E. T. Copson.*

Gould, R. N. and Cunliffe, A. An extended use of perturbation theory. Philos. Mag. (7) 45, 818–822 (1954).

A procedure is given for finding the solution of the partial differential equation $(H+U)\varphi=0$ in terms of the (assumed known) solution of $H\theta=0$ and the eigenfunctions of $H\psi=\lambda\psi$. The boundary conditions for θ are taken to be the same as those for φ, and ψ is required to vanish on the boundary. Expressing U linearly in terms of some parameters, it is assumed that φ can be expanded in terms of these parameters. The successive approximations to φ are then found from a sequence of differential equations, usually by means of an eigenfunction expansion method. An example is given up to the first order. *T. E. Hull* (Vancouver, B. C.).

Blum, E. K. The solutions of the Euler-Poisson-Darboux equation for negative values of the parameter. Duke Math. J. 21, 257–269 (1954).

The objective of the present paper is the determination of all solutions of the hyperbolic differential equation:

$$E(k):\quad\frac{\partial^2u}{\partial x^2}-\frac{\partial^2u}{\partial t^2}-\frac{k}{t}\frac{\partial u}{\partial t}=0$$

for a given negative k. The Cauchy problem with which the author is concerned can be formulated as follows. Let (B, C) be an open interval of the x-axis. Let T be a characteristic triangle in the (x, t)-plane formed by the closed segment B, C and the characteristics which pass through points B and C. Further let G be a region which contains at least the points in the interior of T as well as those on sides AB and AC. A solution $u^{(k)}$ of $E(k)$ is "regular in T" if there is a region G of the above type in which $u^{(k)}$ has continuous second derivatives. By a solution of the Cauchy problem for $E(k)$ the author means a solution $u^{(k)}(t, x)$ of the equation which is regular in T and such that

(1) $\lim_{t\to0^+}u^{(k)}(x, t)=f(x)$ and (2) $\lim_{t\to0^+}u_t^{(k)}(x, t)=0$.

$f(x)$ is an arbitrary function having continuous derivatives of sufficiently high order. Then the following theorems are proved. (1) If $u(x, t)$ is a solution of $E(1)$ which is regular in T, then there are two functions, $\varphi(x)$ and $\psi(x)$, each having continuous first derivatives in (B, C) and such that, for points inside T, u is given by the Poisson integral formula

$$u(x, t)=\int_0^1\varphi[x+(1-2\alpha)t]\alpha^{-\frac{1}{2}}(1-\alpha)^{-\frac{1}{2}}d\alpha+\int_0^1\psi[x+(1-2\alpha)t]\alpha^{-\frac{1}{2}}(1-\alpha)^{-\frac{1}{2}}\lg[t\alpha(1-\alpha)]d\alpha.$$

(2) If $\varphi(x)$ and $\psi(x)$ are any two functions having continuous second derivatives in (B, C), then the function (*) is a solution of $E(1)$ regular in T. (3) If $u^{(-1)}$ is a solution of $E(-1)$ regular in T, then there exist two functions, $\varphi(x)$ and $\psi(x)$ having continuous second derivatives in (B, C) and such that for any (x, t) in triangle T, $u^{(-1)}$ is given by

$$u^{(-1)}(x, t)=t\int_0^1\varphi'[x+(1-2\alpha)t]\alpha^{-\frac{1}{2}}(1-\alpha)^{-\frac{1}{2}}(1-2\alpha)d\alpha+t\int_0^1\psi'[x+(1-2\alpha)t]\alpha^{-\frac{1}{2}}(1-\alpha)^{-\frac{1}{2}}(1-2\alpha)\lg[t\alpha(1-\alpha)]d\alpha+\int_0^1\psi[x+(1-2\alpha)t]\alpha^{-\frac{1}{2}}(1-\alpha)^{-\frac{1}{2}}d\alpha.$$

(4) If $f(x)$ is an arbitrary function having continuous derivatives of order $n+3$ on the interval (B, C), then every solution $u^{(-k)}$ of the Cauchy problem for $E(-k)$, where $k=2n+1$, $n=0, 1, 2, \cdots$, is of the form

$$(**)\qquad u^{(-k)}(x, t)=\sum_{r=1}^{n+1}a_{r,n+1}t^r\frac{\partial^r\Psi}{\partial t^r}+t^{1+k}u^{(2+k)}(x, t),$$

where the function $\Psi(x)$ is determined by the equation

$$f(x)=\pi\left[a_{1,n+1}+\sum_{r=2}^{n+1}a_{r,n+1}(r-1)(-1)^{r+1}\right]\Psi(x)$$

and $u^{(2+k)}$ is a solution of $E(2+k)$ regular in T and such that $\lim_{t\to0}t^nu^{(2+k)}=0$, $\lim_{t\to0}t^{n+1}u_t^{(2+k)}=0$. (5) If the solution $u^{(-k)}$, $k=2n+1$, has continuous derivatives of order

$n+3$ in the interior of triangle T, then $\lim_{t\to 0} u^{(2n+3)}$ is finite and continuous on (B, C) and $\lim_{t\to 0} u_t^{(2n+3)}=0$; i.e., $u^{(2n+3)}$ in (**) is a solution of a Cauchy problem for $E(2n+3)$. (6) A necessary condition that the Cauchy problem for $E(-2n-1)$ have a solution is that the given function $f(x)$ must have derivatives of order at least $n+2$. (7) If $f(x)$ has continuous derivatives of order $n+2$, then every solution $u^{(-2n)}$ of the Cauchy problem for $E(-2n)$ is of the form

$$u^{(-2n)}(x,t)=\sum_{r=1}^{n+1} a_{r,n+1}t^{r-1}\frac{\partial^{r-1}u^{(0)}}{\partial t^{r-1}},$$

where $u^{(0)}(x,t)$ is a regular solution of the wave equation having continuous derivatives of order $n+2$ and such that $u^{(0)}(x,0^+)=f(x)/a_{1,n+1}$. (8) Every solution of the Cauchy problem for $E(-k)$, $k>0$ and not an integer, is given by

$$(***)\quad u^{(-k)}(x,t)=\sum_{r=1}^{n}\sum_{j=0}^{r} a_{rn}\frac{\Gamma(2-s)}{\Gamma(2-s-j)}\binom{r}{j}t^{r-j}\frac{\partial^{r-j}\Phi(s)}{\partial t^{r-j}}+\sum_{r=1}^{n} a_{rn}t^{s+r-1}\frac{\partial^r\omega}{\partial t^r},$$

$$\omega=\int_0^1 \psi[x+(1-2\alpha)t]\alpha^{s-1}(1-\alpha)^{s-1}d\alpha$$

with $\varphi(x)$ determined by

$$u^{(-k)}(x,0^+)=\varphi(x)\sum_{r=1}^{n} a_{rn}\frac{\Gamma^2(1-\frac{1}{2}s)}{\Gamma(2-s-r)}=f(x).$$

The second term in (***) is of the form $t^{1+k}u^{(2+k)}$, where $u^{(2+k)}$ is a solution of $E(2+k)$ such that

$$\lim_{t\to 0} t^{n-1}u^{(2+k)}=0 \quad (2+k=2n+s;\ 0<s<2;\ s\neq 1;\ n=1, 2, \cdots).$$

M. Pinl (Cologne).

Blum, E. K. The Euler-Poisson-Darboux equation in the exceptional cases. Proc. Amer. Math. Soc. **5**, 511–520 (1954).

The Cauchy problem for the Euler-Poisson-Darboux equation requires finding a function $u(x_1, \cdots, x_m, t)$ which satisfies the three conditions

$$L_k[u]\equiv \Delta u-u_{tt}-kt^{-1}u_t=0,$$
$$u(x_1, \cdots, x_m, 0)=f(x_1, \cdots, x_m),$$
$$u_t(x_1, \cdots, x_m, 0)=0; \quad -\infty<k<+\infty.$$

For $k=m-1$, the solution has been given by L. Åsgeirsson [Math. Ann. **113**, 321–346 (1936)]. For $k<m-1$, but $k\neq -1, -3, -5, \cdots$, the solution has been given by A. Weinstein in the paper reviewed below. Another solution for $k<m-1$ is obtained by Diaz and Weinberger [Proc. Amer. Math. Soc. **4**, 703–715 (1953); these Rev. **15**, 321] including the exceptional cases $k=-1, -3, -5, \cdots$. In this paper the author obtains by essentially different methods another (explicit) solution for the exceptional cases.

His treatment of the exceptional cases is based on the two fundamental identities

$$u^k(x,t)=t^{1-k}u^{(2-k)}(x,t), \quad u_t^k(x,t)=t^{-1}u^{(k+2)}(x,t)$$

given by Darboux. These identities can be rewritten by introducing the variable $s=t^2$ as

$$(*)\quad u^k(x,s^{1/2})=s^{(1-k)/2}u^{2-k}(x,s^{1/2}),$$
$$2u_s^k(x,s^{1/2})=u^{(k-2)}(x,s^{1/2}).$$

Using (*) the problem for the exceptional cases can be reduced to the case $k=1$. The author shows that for the exceptional values of k, a small deviation, no matter how regular, from polyharmonic initial values introduces logarithmic infinites in certain derivatives of the solution.

M. Pinl (Cologne).

***Weinstein, Alexander. On the wave equation and the equation of Euler-Poisson.** Proceedings of Symposia in Applied Mathematics, Vol. V, Wave motion and vibration theory, pp. 137–147. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954. $7.00.

The author intends to give a solution of Cauchy's problem for the hyperbolic equation

$$(*)\quad \Delta u^k=u^k_{tt}+kt^{-1}u^k_t,$$

where the superscript k indicates the dependence of u on the parameter k. The initial conditions are

$$u^k(x,0)=f(x_1, \cdots, x_m), \quad u_t^k(x,0)=0,$$

for a given sufficiently regular function f. The differential equation (*) is called the equation of Euler-Poisson-Darboux (for all positive integers m and all real values of k). There are two fundamental recursion formulae:

$$u_t^k(x,t)=tu^{k+2}(x,t); \quad u^k(x,t)=t^{1-k}u^{2-k}(x,t),$$

which allow one to obtain from a solution u^k of (*) solutions of the same equation with the parameter $k+2$ and $2-k$ respectively. Cauchy's problem for the Euler-Poisson-Darboux equation for the case $k=m-1$ has been solved explicitly in the book of Courant and Hilbert [Methoden der mathematischen Physik, vol. II, Springer, Berlin, 1937, p. 411]. The cases $k=m, m+1, \cdots$ can be solved by the method of descent introduced by Hadamard. This method of descent can be generalized for all (non-integral) values of k which are greater than $m-1$. In the case $k<m-1$ but $k\neq -1, -3, -5, \cdots$ the Cauchy problem can be solved by the same method and repeated application of the fundamental recursion formulas, starting with the function $u^{k+2n}(x,t)$, $k+2n\geqq m-1$. If $m-k$ is a positive odd integer, then n can be chosen so that $k+2n=m-1$. This and only this case corresponds to a Huygens principle as already noted by J. B. Diaz and H. F. Weinberger [Proc. Amer. Math. Soc. **4**, 703–715 (1953); these Rev. **15**, 321]. For $k>m-1$ and $k\geqq m-1-2n$, $k\neq -1, -3, -5, \cdots$ the solution can be represented by the Riemann-Liouville integral. If $k=0, 1, 2, \cdots$ the Euler-Poisson-Darboux equation can be considered as a Coulon, or ultrahyperbolic equation. The case $k=0$ has been considered by Asgeirsson (n is supposed to be radially symmetric in Coulon's equation). The author does not use Abel's integral equation or analytic continuations in this connection. Finally the exceptional cases $k=-1, -3, -5, \cdots$ are discussed and the uniqueness of the solutions.

M. Pinl (Cologne).

Weinstein, Alexander. The singular solutions and the Cauchy problem for generalized Tricomi equations. Comm. Pure Appl. Math. **7**, 105–116 (1954).

Using Chaplygin's notation, the Tricomi equation may be written in the form

$$\sigma\psi_{ss}+\psi_{vv}=0.$$

For $\sigma>0$ the equation is elliptic and may be written

$$u_{xx}+u_{yy}+\frac{1}{3y}u_y=0, \quad x=\theta, \quad y=\tfrac{2}{3}\sigma^{3/2}.$$

For $\sigma<0$ the equation is hyperbolic and may be written

$$u_{xx}-u_{tt}-\frac{1}{3t}u_t=0, \quad x=\theta, \quad t=\tfrac{2}{3}(-\sigma)^{3/2}.$$

Generalizations are

$$\sigma^{2n+1}\sum_{i=1}^{m}\psi_{\theta\theta_i}+\psi_{\sigma\sigma}=0$$

and

$$\sum_{i=1}^{m}u_{x_ix_i}+u_{yy}+\frac{k}{y}u_y=0,$$

$$(*)\qquad k=(2n+1)(2n+3)^{-1},\quad n=0,1,2,\cdots.$$

$$\sum_{i=1}^{m}u_{x_ix_i}-u_{tt}-\frac{k}{t}u_t=0,$$

In order to emphasize the dependence of a solution of (*) on the index the author denotes such a solution by $u^{(k)}(x,y)$ and $u^{(k)}(x,t)$ respectively. Then the relations

$$(**)\quad u^{(k)}(x,t)=t^{1-k}u^{(2-k)}(x,t),\quad u_t{}^{k}(x,t)=t^{-1}u^{(k+2)}(x,t)$$

hold, which have been proved first by G. Darboux in the case $m=1$. The equation

$$\sum_{i=1}^{m}u_{x_ix_i}+u_{yy}+\frac{k}{y}u_y=0$$

may be regarded as a generalization of the classical Laplace equation in the axially symmetric case. The author gives two fundamental solutions in the large using the correspondence principle $u^{(k)}=y^{1-k}u^{(2-k)}$.

The first fundamental solution $u_b{}^{(k)}$ is continuous for all values $y\geqq 0$, analytic for $y>0$ and has an appropriate singularity at a single point, say $x=0$, $y=b>0$. Then $u_b{}^{(k)}$ is a fundamental solution in the sense of Hadamard, vanishing as x^2+y^2 tends to infinity. The second fundamental solution $v_b{}^{(k)}=y^{1-k}u_b{}^{(2-k)}$ vanishes for $y=0$ and at infinity, but is not analytic in $y=0$ except for $k=0,-1,-2,\cdots$ and is a solution in the large for $k<1$. This solution was also given by Olevskiĭ in terms of hypergeometric series [Doklady Akad. Nauk SSSR, (N.S.) **64**, 767–770 (1949); these Rev. **10**, 609]. To continue these solutions into the hyperbolic half-plane ($\sigma\leqq 0$), the singular Cauchy problem for the Euler-Poisson-Darboux equation

$$\sum_{i=1}^{m}\frac{\partial^2 u}{\partial x_i{}^2}=\frac{\partial^2 u}{\partial t^2}+\frac{k}{t}\frac{\partial u}{\partial t},\quad -\infty<k<+\infty,$$

with the initial conditions

$$u(x_1,\cdots,x_m,0)=f(x_1,\cdots,x_m),\quad \text{or}\quad u(x,0)=f(x),$$
$$u_t(x_1,\cdots,x_m,0)=0,\quad \text{or}\quad u_t(x,0)=0,$$

must be solved.

For $k=m-1$ the solution of the Cauchy problem is given by Asgeirsson's mean-value formula. This formula reduces to Poisson's formula for $m=3$. For $k>m-1$ the solution is given by a formula which coincides with the formula obtained by Hadamard's method of descent for $k=m$, $m+1,\cdots$. This principle of obtaining a solution for integral values of $k>m-1$ and extending it to non-integral values may be regarded as a generalized method of descent. To solve the Cauchy problem $k<m-1$, $k\neq -1,-3,-5,\cdots$, the author makes repeated use of the formulas (**). The results of the generalized method of descent can also be represented by the Riemann-Liouville integral. For $k=0,1,2,3,\cdots$ the Euler-Poisson-Darboux equation can be considered as a Coulon, or ultra-hyperbolic equation. The case $k=0$ has been considered by Asgeirsson. The author assumes that u in Coulon's equation is radially symmetric in $y_1,\cdots,y_{k+1}$, while Asgeirsson assumes that u depends only on y_1 which he identifies with t. Regarding the solutions for $k\neq -1,-3,-5,\cdots$, the only assumption on $f(x_1,\cdots,x_m)$ was that $f(x_1,\cdots,x_m)$ should have continuous derivatives of order greater than or equal to $\frac{1}{2}(m+3-k)$. In that case, the solution also has continuous derivatives of the same order with respect to t, even for $t=0$. In the exceptional cases $k=-1,-3,-5,\cdots$ a solution $u^{(k)}$ exists under the same differentiability conditions on f. However, the partial derivative $\partial^{1-k}u^{(k)}/\partial t^{1-k}$ of this solution becomes infinite like $\lg t$ at $t=0$ unless $f(x_1,\cdots,x_m)$ is polyharmonic of order $\frac{1}{2}(1-k)$. In this generality, a solution was found by Diaz and Weinberger and Blum. It is remarkable that for the exceptional values even a small deviation, no matter how regular, from polyharmonic initial values introduces logarithmic infinities in certain derivatives of the solution.

Finally, the author exemplifies his results by giving explicitly the first fundamental solution of the generalized Tricomi equation for $\sigma\leqq 0$ and $\sigma\geqq 0$. *M. Pinl.*

Gårding, Lars. L'inégalité de Friedrichs et Lewy pour les équations hyperboliques linéaires d'ordre supérieur. C. R. Acad. Sci. Paris **239**, 849–850 (1954).

Soit l'opérateur

$$a\left(x,\frac{\partial}{\partial x}\right)=\sum_{\Sigma\alpha\leqq m+1}a_{\alpha_1\cdots\alpha_n}(x_1,\cdots,x_n)\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n},$$

hyperbolique d'un ouvert $\Omega\subset R^n$, où $a_{\alpha_1\cdots\alpha_n}(x)\in C_1$ si $\sum\alpha=m+1$, $\in C_0$ sinon. Dans une région lenticulaire $T_\lambda\subset\Omega$, limitée par les surfaces S_0, fixe, et S_λ, $\lambda\in\,]0,\lambda_0[$, variable, suffisamment régulières et d'orientation d'espace, il existe une constante c indépendante de $u\in C_{m+1}(\Omega)$ et de λ, telle que

$$c\int_{S_\lambda}\sum_{\Sigma\alpha\leqq m}\left|\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}u\right|^2 dS_\lambda$$
$$\leqq\int_{S_0}\sum_{\Sigma\alpha\leqq m}\left|\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}u\right|^2 dS_0+\int_{T_\lambda}|au|^2dT_\lambda.$$

La démonstration est seulement esquissée. L'inégalité obtenue, généralisation de celle de Friedrichs et Lewy [Math. Ann. **98**, 192–204 (1927)], intervient dans la théorie du problème de Cauchy relatif à $a(x,\partial/\partial x)$. *H. G. Garnir.*

Manaresi, Fabio. Applicazione di un procedimento variazionale allo studio di una equazione differenziale alle derivate parziali con caratteristiche reali doppie. Rend. Sem. Mat. Univ. Padova **23**, 163–213 (1954).

By the use of variational methods the author establishes the existence of a unique solution of

$$(\theta(x,y)u_{xy})_{xy}+p(x,y)u=0,\quad (x,y)\text{ on }R,$$
$$u=g(x,y),\quad (x,y)\text{ on boundary }FR\text{ of }R,$$

where R is the rectangular region $a_1\leqq x\leqq a_2$, $b_1\leqq y\leqq b_2$, the continuous function $p(x,y)$ is non-negative and $\theta(x,y)$ is positive and possesses suitable differentiability properties on R, while $g(x,y)$ is continuous and satisfies certain differentiability conditions on FR. The second portion of the paper deals with the determination of proper values and proper functions of

$$(\theta(x,y)u_{xy})_{xy}+[p(x,y)-\lambda q(x,y)]=0,\quad (x,y)\text{ on }R,$$
$$u=0,\quad (x,y)\text{ on }FR,$$

where $\theta(x,y)$, $p(x,y)$ satisfy the conditions previously mentioned, and $q(x,y)$ is a non-identically vanishing continuous function on R. Indication is given as to how certain details of proof may be modified to obtain corresponding results for a somewhat more general type of differential equation. *W. T. Reid* (Evanston, Ill.).

***Leray, Jean. Hyperbolic differential equations.** The Institute for Advanced Study, Princeton, N. J., 1953. 240 pp. (mimeographed). $4.00.

Ce livre est consacré au problème de Cauchy pour les équations aux dérivées partielles. Il constitue une contribution des plus marquantes dans ce domaine par l'originalité de l'exposé et l'intérêt des résultats nouveaux qu'il contient.

La première partie (pp. 1–103) concerne les équations aux dérivées partielles linéaires et à coefficients constants. L'auteur édifie d'abord un calcul symbolique à plusieurs variables. Il commence par définir la transformée de Laplace $\mathfrak{L}\mathcal{C}$ des distributions $\mathcal{C} \in \mathfrak{D}(R^n)$ qui sont des sommes finies de dérivées de fonctions $f(x)$, $x \equiv x_1, \cdots, x_n$, telles que $\exp(-\Sigma_1^n x_i\xi_i)f \in L^2(R^n)$ pour $\xi \in \Omega$, domaine convexe donné. Les transformées de Laplace $\mathfrak{L}\mathcal{C}$ sont des fonctions $a(\zeta)$, $\zeta = \xi + i\eta$, sommes finies de produits de polynômes en ζ par des fonctions $b(\zeta)$ analytiques, régulières dans le tube de base Ω et telles que $\sup_\eta |b(\xi+i\eta)|$ soit uniformément borné dans Ω. Cela étant, l'auteur nomme produit symbolique de $\mathcal{C}$ par $a(\zeta)$ la distribution $a(p)\mathcal{C} = \mathfrak{L}^{-1}a(\zeta)\mathfrak{L}\mathcal{C}$, qui s'avère être $\mathfrak{L}^{-1}a(\zeta) * \mathcal{C}$; il étudie les propriétés de ce produit symbolique et, en particulier, détermine explicitement pour $a(\zeta)$ donné un fermé convexe C tel que la connaissance de $\mathcal{C}$ dans l'ouvert $\mathcal{O} - C$ détermine $a(p)\mathcal{C}$ dans $\mathcal{O}$, ouvert quelconque de R^n.

Toutes ces considérations sont basées sur la théorie de la transformation de Fourier et de Laplace dans $L^2(R^n)$. Les résultats sont parallèles à ceux obtenus par L. Schwartz [Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] Tome Supplémentaire, 196–206 (1952); ces Rev. 14, 639] et J. L. Lions [J. Analyse Math. 2, 369–380 (1953); ces Rev. 15, 307] dans le cadre de la théorie des distributions.

La distribution $\mathfrak{L}^{-1}a(\zeta)$ est une solution élémentaire au sens de L. Schwartz de l'opérateur linéaire à coefficients constants $a(\partial/\partial x)$, $\partial/\partial x \equiv \partial/\partial x_1, \cdots, \partial/\partial x_n$. Elle permet de résoudre, par composition, le problème de Cauchy pour cet opérateur, formulé dans la théorie des distributions. L'étude de cette solution élémentaire exige l'introduction de l'ouvert Δ, intérieur de l'ensemble des points ξ pour lesquels $a(\xi+i\eta)$ diffère de zéro quel que soit η. La solution élémentaire étudiée possède une forme différente, indépendante de ξ, lorsque ξ appartient à chaque composante connexe (convexe) de Δ; son support est alors le cône réciproque du cône directeur de la composante connexe en question. La considération de l'ouvert Δ et le théorème relatif au support de la solution élémentaire permettent d'obtenir d'une manière très directe (et même de compléter) les résultats de L. Gårding [Acta Math. 85, 1–62 (1951); ces Rev. 12, 831].

L'auteur développe enfin les résultats annoncés dans une note antérieure [C. R. Acad. Sci. Paris 234, 1112–1115 (1953); ces Rev. 14, 477]. Lorsque $a(p)$ est homogène, on sait que la solution élémentaire s'exprime au moyen de périodes d'intégrales abéliennes [cf. Herglotz, Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Phys. Kl. 78, 93–126, 287–318 (1926); 80, 69–114 (1928); I. Petrowsky, Mat. Sbornik N.S.17(59), 289–370 (1945); ces Rev. 8, 79; et articles cités par F. Bureau dans Colloque de Géométrie algébrique, Liège, 1949, Thone, Liège, 1950, pp. 155–176; ces Rev. 12, 338]. L'auteur donne à ces calculs une forme achevée et rigoureuse; il ne suppose plus que le cône $a(\xi) = 0$ n'a pas de points singuliers; il met les formules finales sous une forme invariante, valable quel que soit l'ordre de l'équation; il précise également un critère de I. Petrowsky [loc. cit.] concernant l'existence de lacunes stables.

Cette première partie de l'ouvrage contient de nombreux exemples; en particulier, l'auteur explicite tous ses résultats dans le cas de l'opérateur des ondes.

Les équations aux dérivées partielles hyperboliques, linéaires et à coefficients variables font l'objet de la deuxième partie du livre (pp. 104–219). L'auteur établit d'abord des inégalités applicables à la solution du problème de Cauchy pour l'opérateur $a(x, \partial/\partial x)$ (polynôme en $\partial/\partial x_1, \cdots, \partial/\partial x_n$ dont les coefficients sont fonctions de $x_1, \cdots, x_n$) dans le voisinage de la surface qui porte les données initiales. La méthode utilisée prend comme point de départ une égalité analogue à celle de l'énergie, obtenue dans le cas général après d'intéressantes considérations algébriques; l'opérateur $a(x, \partial/\partial x)$ est supposé hyperbolique dans le domaine considéré, c'est-à-dire qu'en chaque point x de ce domaine, le cône $a_m(x, p) = 0$ ($a_m(x, p)$, termes de degré maximum de $a(x; p)$) contient des points p par lesquels toute sécante réelle coupe le cône en des points réels et distincts. Utilisant les inégalités obtenues, l'auteur étend la solution locale de proche en proche et obtient ainsi une preuve complète de l'affirmation de I. Petrowsky [Mat. Sbornik N.S. 2(44), 815–870 (1937)] disant que le problème de Cauchy admet toujours une solution globale lorsque l'équation est hyperbolique au sens signalé (avec quelques restrictions trop longues à indiquer ici). A cette occasion, il soumet le mémoire de Petrowsky à une critique sévère.

Une fois démontrée l'existence de la solution globale, l'auteur donne de nouvelles inégalités plus simples, relatives à cette solution. Il les obtient sous des hypothèses plus générales sur les coefficients de l'équation, qu'il suppose bornés et lipschitziens au lieu d'en postuler toutes les dérivées bornées. Grâce à ces nouvelles inégalités, l'auteur introduit les opérateurs inverses de $a(x, \partial/\partial x)$ qui permettent de résoudre le problème de Cauchy dont les données initiales sont nulles à l'infini. L'auteur fait également une étude complète du domaine de dépendance de la solution du problème de Cauchy. A cette occasion, il reprend et complète des travaux de A. Marchaud [Compositio Math. 3, 89–127 (1936); Bull. Sci. Math. (2) 62, 229–240 (1938)] et S. C. Zaremba [ibid. 60, 139–160 (1936)] sur les fronts d'émission par un ensemble donné. Il montre que les fronts d'émission qui limitent les domaines de dépendance sont le lieu de trajectoires définies par les équations de Jacobi, de Lagrange ou de Hamilton issues d'un principe variationel.

Toutes les considérations précédentes sont enfin étendues aux opérateurs hyperboliques sur une variété ou aux systèmes linéaires hyperboliques à coefficients variables.

La troisième partie du livre (pp. 220–238) est relative aux équations et systèmes d'équations hyperboliques non linéaires. On y trouve seulement un théorème local d'existence et un théorème global d'unicité. Contrairement à I. Petrowsky [Mat. Sbornik N.S. 2(44), 815–870 (1937)] l'auteur utilise une méthode d'approximations successives inspirée des travaux de J. Schauder pour les équations du second ordre [Fund. Math. 24, 213–246 (1935)].

H. G. Garnir (Liège).

Ciliberto, Carlo. Formule di maggiorazione e teoremi di esistenza per le soluzioni delle equazioni paraboliche in due variabili. Ricerche Mat. 3, 40–75 (1954).

Let T be the closed region $0 \leq x \leq X$, $0 \leq y \leq Y$, and let T' denote the boundary points of T which lie on the three lines $x=0$, $y=0$, and $x=X$. Assume (x, y), (x', y') are points of T and form for the function $z(x, y)$ the ratio $|z(x, y) - z(x', y')| / (|x-x'|^{\mathfrak{a}} + |y-y'|^{\lambda})$. In case this ratio

is bounded in T the symbol $|z|_\lambda^T$ is used to indicate its least upper bounds, and the function z will be said to belong to class $H^{(\Omega,\lambda)}$. Here λ is a fixed positive constant less than or equal to $\frac{1}{2}$. The present paper is devoted mainly to obtaining estimates of such numbers as $|u|_\lambda^T$, $|u_x|_\lambda^T$, $\cdots$, where u is a solution of the parabolic equation (*) $u_{xx}-u_y=f(x,y)$ or of the more general equation

(**) $u_{xx}+a_1(x,y)u_y+a_2(x,y)u_x+a_3(x,y)u=f(x,y)$.

The following theorem is typical of the results obtained. If $a_1<0$, a_2, a_3, f are functions of class $H^{(\Omega,\lambda)}$, then there exists a constant C dependent only on T and a_1, a_2, a_3 such that for every solution u of (**) interior to T and vanishing on T' we have

$$|u_{xx}|_\lambda^T+|u_y|_\lambda^T<C[\max_T|f|+|f|_\lambda^T].$$

As an immediate consequence of theorems of this type the author obtains the following existence theorem. As before let $a_1<0$, a_2, a_3, f be of class $H^{(\Omega,\lambda)}$; then there exists a function $u=u(x,y)$ which interior to T is a solution of (**) and which on T' takes on preassigned continuous values. The methods used by M. Gevrey [C. R. Acad. Sci. Paris **193**, 690–692, 1061–1063 (1932)] to obtain this result are quite different from those used in the present paper.

F. G. Dressel (Durham, N. C.).

Pucci, Carlo. Studio col metodo delle differenze di un problema di Cauchy relativo ad equazioni a derivate parziali del secondo ordine di tipo parabolico. Ann. Scuola Norm. Super. Pisa (3) **7** (1953), 205–215 (1954).

Let $a(t)$, $b(t)$, $c(t)$, $f(x,t)$ be continuous functions on the rectangle $R(0\leqq x\leqq l,\ 0\leqq t\leqq h)$, and let the derivatives of the functions $u_0(x)$, $u_1(x)$, $f(x,t)$ on R satisfy the inequalities

$$\left|\frac{d^nu_0}{dx^n}\right|,\quad \left|\frac{d^nu_1}{dx^n}\right|,\quad \left|\frac{\partial^nf}{\partial x^n}\right|<M(2n)!/g^n,\quad n=1,2,\cdots.$$

Here M and g are positive constants with $g>l^2h^2\max|a(t)|$. Using difference methods the author proves that there exists a function $u(x,t)$, which in R has continuous partial derivatives $\partial^iu/\partial t^i$, $\partial^nu/\partial x^n$ $(i=1,2;\ n=1,2,\cdots)$ and which is a solution of the following Cauchy problem:

$$u_{tt}=a(t)u_{xx}+b(t)u_t+c(t)u+f(x,t),\quad (x,t)\text{ in }R,$$
$$u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x),\quad 0\leqq x\leqq l.$$

F. G. Dressel (Durham, N. C.).

Difference Equations, Special Functional Equations

Jost, Res. Lineare Differenzengleichungen mit periodischen Koeffizienten. Comment. Math. Helv. **28**, 173–185 (1954).

The difference equation $\sum_{k=0}^r a_k(z)H(z+k)=0$ is investigated in the case that the coefficients $a_k(z)$ are meromorphic functions with period unity. More particularly it is assumed that the coefficients are rational functions of $v=e^{2\pi iz}$ and that the polynomial $P(\lambda)=\sum_{k=0}^r a_k(z)\lambda^k$ is irreducible in the field of rational functions of v. By multiplication by a suitable polynomial in v, the original equation may be written in the form $\sum_{k=0}^r p_k(v)H(z+k)=0$, where the coefficients $p_k(v)$ are relatively prime polynomials in v. The singular points of the equation are defined to be the zeros of $p_0(e^{2\pi iz})$ and $p_r(e^{2\pi iz})$. It is first proved that if $H(z)$ is a solution of the equation, the functions $H(z)$, $H(z+1)$, $\cdots$, $H(z+r-1)$ constitute a fundamental set of solutions. The author then constructs a solution $H(z)$ which is meromorphic and possesses poles only at singular points of the equation.

The foregoing results are extended to the case of coefficients with period an arbitrary integer $m\geqq 1$. In particular, meromorphic solutions are obtained for the system of equations $Y(z+1)=M(z)Y(z)$ where $M(z)$ is a non-singular matrix with elements which are rational functions of $u=e^{(2\pi iz/m)}$.

P. E. Guenther (Cleveland, Ohio).

Minorsky, Nicolas. Sur quelques applications des équations différentielles aux différences. Rend. Sem. Mat. Fis. Milano **23** (1952), 36–47 (1953).

The author first singles out for particular discussion the equation

$$\frac{d^2x(t)}{dt^2}+a\frac{dx(t)}{dt}+\lambda x(t-h)=0 \tag{1}$$

and its characteristic equation

$$z^2+az+\lambda e^{-hz}=0. \tag{2}$$

where h and a are positive and λ real. He is concerned with the variation of the roots of (2), or, what is the same thing, the changes in the frequency and amplitude of the solutions of (1) as the parameter λ is varied. λ might, for example, be the feedback in a control system with a lag. Such a system will, as the author shows, first not oscillate if λ is very small and positive; as λ is increased it will reach a series of thresholds at each of which a new frequency is added to the undamped solutions of (1). He shows how to find the critical values of λ.

As the author observes, oscillations with indefinitely increasing amplitude cannot occur in a physical system, so that the above conclusion makes no physical sense. Accordingly he is led to discuss the equation

$$\frac{d^2x(t)}{dt^2}+a\frac{dx(t)}{dt}+\lambda[x(t-h)+\epsilon(x(t-h))^3]=0,$$

which he studies by a method developed by himself elsewhere [C. R. Acad. Sci. Paris **232**, 2179–2180 (1951); these Rev. **13**, 38]. These can have stable physically sensible oscillations, and he gives a full discussion of them. The details are too complicated to sketch here.

J. M. Danskin (Washington, D. C.).

Hahn, Wolfgang. Bericht über Differential-Differenzengleichungen mit festen und veränderlichen Spannen. Jber. Deutsch. Math. Verein. **57**, Abt. 1, 55–84 (1954).

Expository paper.

Ishiguro, Kazuo. Sur le problème de l'équation fonctionnelle. Tôhoku Math. J. (2) **6**, 1–4 (1954).

Let the real measurable function $\theta(x)$ have the property that to every decreasing sequence of numbers ϵ_k with $\epsilon_k\to 0$ there corresponds a sequence of real numbers η_k with $\eta_k\to 0$, such that

$$\lim_{k\to\infty}\text{meas}\,\{x\,|\,\theta(x+\epsilon_k)\neq\eta_k+\theta(x)\}=0.$$

Then $\theta(x)$ is linear on a set of positive measure. The latter part of the proof is expressed geometrically, and the author concludes with a simpler (analytic) demonstration of that part suggested by Shigeki Yano.

I. M. Sheffer.

Heinhold, J. Zur Lösung gewisser Funktionalgleichungen. Arch. Math. **5**, 414–422 (1954).

Let fg stand for $f(g(x))$, and similarly for more "factors". The functional equation (1) $fhfh\cdots hf=ghgh\cdots hg$ is

treated, where the left side contains n f's and the right side m g's, and either (a) g, h are given and f sought, or (b) f, g given and h sought. The variable x and all functions are real. As to (a): Assume $g(x)$, $h^{-1}(x)$ are continuous and both monotonic increasing [both decreasing] for all x, with $g(x)>h^{-1}(x)$. To each set $\mathfrak{A}_n$ of numbers x_r such that $x_0<\cdots<x_{n-1}<x_n=h(g(x_0))$, there is associated, by a graphical method involving step-polygons with vertices on the curves (*) $y=g(x)$, $y=h^{-1}(x)$, a denumerable set X_n. It is shown that there is a solution $y=f(x)$ of (1) for x restricted to the set X_n. By varying the set $\mathfrak{A}_n$ continuously, solutions of (1) continuous for all x are obtained; and every solution, continuous and monotonic increasing, and lying in the strip determined by the curves (*), is obtainable by this method. Problem (b) is similarly handled under the assumptions that f, g are continuous and monotonic increasing and $g(x)>f(x)$. *I. M. Sheffer.*

Urabe, Minoru. Equations of Schröder. J. Sci. Hiroshima Univ. Ser. A. **15**, 113–131 (1951), 203–233 (1952).

Let $\varphi^\mu(x)$ $(\mu=1,\cdots,n)$ be functions of the n variables $x=(x_1,\cdots,x_n)$, regular about $x=0$; $\varphi^\mu(x)=a_\nu{}^\mu x^\nu+$higher powers, with $\det A\neq 0$ $(A=\|a_\nu{}^\mu\|)$. Let λ_i $(i=1,\cdots,n)$ be the eigenvalues of A. Under the assumption (*) $|\lambda_i|<1$ (all n) Fukuhara [Kyushu Teikoku Daigaku, Rigaku Hokoku **1**, no. 2, 189–196 (1945)] proved that the following generalized Schröder equations have a solution regular about $x=0$:

$$(S)\qquad f^i{}_{lp}[\varphi(x)]=\lambda_i f^i{}_{lp}(x)+f^i{}_{lp-1}(x)+\Psi^i{}_{lp}(x).$$

Here $\Psi^i{}_{lp}$ is a suitably chosen linear combination of certain polynomials in the unknown functions $f^i{}_{mq}$, and each of i, l, p ranges over a fixed set of positive integers determined by the matrix A. The author obtains a solution of (S) by showing first that power series exist for the $f^i{}_{lp}$ that formally satisfy (S) (for a unique choice of the multipliers in the Ψ's), and then that these series converge. (This holds also for the case $|\lambda_i|>1$.)

Now consider the transformation $\mathfrak{F}$:

$$'x^\mu=\varphi^\mu(x)=a_\nu{}^\mu x^\nu+\cdots\qquad(\det A\neq 0),$$

and suppose $\mathfrak{F}$ can be imbedded in a 1-parameter group $\mathfrak{G}$ of transformations, giving $\mathfrak{F}$ for $t=t_0$, say. Let the operator of the group be $X\equiv\xi^\mu\partial/\partial x^\mu$, ξ^μ assumed regular about $x=0$: $\xi^\mu=c_\nu{}^\mu x^\nu+\cdots$, the finite form of $\mathfrak{G}$ being $'x^\mu=a_\nu{}^\mu(t)x^\nu+\cdots$, obtained from $'x^\mu=e^{tX}(x^\mu)$. It is assumed that the eigenvalues μ_i of $C=\|c_\nu{}^\mu\|$ lie in a convex domain not containing the origin. Set $A(t)=\|a_\nu{}^\mu(t)\|$; then $A(t)=e^{tC}$, and $\lambda_i=e^{t_0\mu_i}$. The functional equation system

$$(S')\qquad f^i{}_{lp}[\varphi]=\lambda_i f^i{}_{lp}(x)+f^i{}_{lp-1}(x)+\Psi^i{}_{lp}(x)$$

is shown to have a solution. Here each Ψ has the same form as the Ψ of (S) but these are not identical. The polynomials in Ψ (of (S)) involve (in a manner too involved to set down here) those non-negative integers $p_1,\cdots,p_R$, with $\sum_1^R p^j\geq 2$, for which there exist relations of the form (a) $\lambda_i=\lambda_1{}^{p_1}\cdots\lambda_R{}^{p_R}$, $\lambda_1,\cdots,\lambda_R$ the distinct eigenvalues of A); and those in Ψ (of (S')) the $p_1,\cdots,p_R$ for which there exist relations (b) $\mu_i=\mu_1 p_1+\cdots+\mu_R p_R$. (b) implies (a) but not conversely, and a condition (necessary and sufficient) is found in order that (a) imply (b).

Given the transformation $\mathfrak{F}$ (as above), from the Schröder system (S) is obtained the existence of a group $\mathfrak{G}$ containing $\mathfrak{F}$; and in the case that (a) and (b) are equivalent, the functions ξ^μ appearing in the group operator are regular about $x=0$. When (a), (b) are not equivalent, conditions are found in order that the ξ^μ be regular about $x=0$. Conditions for the uniqueness of $\mathfrak{G}$ are obtained; and when uniqueness is lacking, the general form of the functions ξ^μ is found. Also, for the given transformation $\mathfrak{F}$, a set of $n-1$ independent invariants is gotten. *I. M. Sheffer.*

Urabe, Minoru. Decomposition of finite transformation into infinitesimal transformations. J. Sci. Hiroshima Univ. Ser. A. **16**, 43–45 (1952).

Given the transformation $\mathfrak{F}$: $'x^\nu=\varphi^\nu(x)=a_\mu{}^\nu x^\mu+\cdots$ $(\det A\neq 0)$; let the eigenvalues of A be λ_i, $i=1,\cdots,n$. If (*) $|\lambda_i|<1$ for all n or $|\lambda_i|>1$ for all n, then [see the preceding review] we have $'x^\nu=e^X(x^\nu)$, where $X\equiv\xi^\mu\partial/\partial x^\mu$ is the operator of a group of transformations including $\mathfrak{F}$. By a simple linear transformation it is shown that if condition (*) does not hold, then $\mathfrak{F}$ can be expressed as the product of two transformations: $\mathfrak{F}=e^{cD}\cdot e^{X_0}(x^\nu)$, where $D\equiv x^\mu\partial/\partial x^\mu$ is the operator of the group of dilations, X_0 corresponds to a group for which ξ^μ are regular at $x=0$, and c is a sufficiently large positive number. *I. M. Sheffer.*

Urabe, Minoru. Invariant varieties for finite transformation. J. Sci. Hiroshima Univ. Ser. A. **16**, 47–55 (1952).

Invariant varieties of the form $\chi^\lambda=f^\lambda(x^\mu)$ are sought for the transformation T: $'x^\nu=\varphi^\nu(x)=a_\mu{}^\nu x^\mu+\cdots$ in the case that the eigenvalues $\{\lambda_i\}$ of $A=\|a_\mu{}^\nu\|$ are not all less than, or all greater than, unity in magnitude [these last cases having been treated earlier; see the two preceding reviews]. By a linear transformation T is put into a canonical form; and from the conditions $f^\lambda[\varphi^\mu(x,f^\mu)]=\varphi^\lambda(x,f^\mu)$ for invariance, formal power series solutions $f^\lambda(x^\mu)$ are obtained, and a fixed-point argument applied to show convergence.

I. M. Sheffer (State College, Pa.).

Urabe, Minoru. Iteration of certain finite transformation. J. Sci. Hiroshima Univ. Ser. A. **16**, 471–486; **17**, 43–65 (1953).

Consider the transformation (1)

$$T:\quad 'x^\nu=\varphi^\nu(x^\mu)=a_\mu{}^\nu x^\mu+a^\nu{}_{\mu_1\mu_2}x^{\mu_1}x^{\mu_2}+\cdots,$$

where the eigenvalues $\{\lambda_i\}$ of $A=\|a_\mu{}^\nu\|$ all satisfy $|\lambda_i|=1$. If all the arguments of the λ_i are commensurable with π, a suitable iteration of T makes all the eigenvalues unity. It is supposed that this is the case here, so that (1) can be brought to canonical form (2)

$$T:\quad 'x^\nu=x^\nu+\delta_{\nu-1}x^{\nu-1}+a^\nu{}_{\mu_1\mu_2}x^{\mu_1}x^{\mu_2}+\cdots.$$

Here $\delta_{\nu-1}=0$ or 1. Transformations T are classified as follows: Type A_1: $\|a_\mu{}^\nu\|$ is of diagonal form and at least one term $a^\nu{}_{\mu_1\mu_2}\neq 0$; Type A_2: $\|a_\mu{}^\nu\|$ not diagonal; Type B: $\|a_\mu{}^\nu\|$ diagonal and all $a^\nu{}_{\mu_1\mu_2}=0$.

Let S be a hypersphere in the space E_{2n} of the n complex numbers $x_1,\cdots,x_n$, of center α^ν and radius r, and passing through the origin. Set $\alpha^\nu=rr_\nu e^{i\omega_\nu}$ $(r_\nu\geq 0,\ \sum_\nu r_\nu{}^2=1)$. Suppose (Condition I) that the hypersurface of every hypersphere lying in S and tangent to S at the origin is carried by T into the interior of the hypersphere (except for the origin). Then by infinite iteration of T, every point of S converges to the origin. Under certain conditions (too detailed to set down) Condition I is satisfied for Type A_1 and Type B transformations; but for Type A_2 there are no such hyperspheres.

In each x^ν-plane separately, draw a circle through the origin, with radius r and center $re^{i\omega_\nu}$. In space E_{2n} these circles determine a cylindrical domain D. Condition II: There is such a D for which every point $P(x^\nu)$ on the boundary of D (except the origin) is carried by T into an inner

point of D. Necessary and sufficient conditions are obtained for Type A_1 and Type B transformations in order that Condition II hold; but for Type A_2 there is no such D. The above results are applied to examine the behavior of the integral curves of the system of differential equations $dx^1/\xi^1=dx^2/\xi^2=\cdots=dx^n/\xi^n$, where the ξ^r are regular at the origin: $\xi^r=c_\lambda{}^r x^\lambda+c^r{}_{\lambda_1\lambda_2}x^{\lambda_1}x^{\lambda_2}+\cdots$, and where it is supposed that the eigenvalues of $\|c_\lambda{}^r\|$ lie on a line through the origin.

I. M. Sheffer (State College, Pa.).

Urabe, Minoru. Application of majorized group of transformations to functional equations. J. Sci. Hiroshima Univ. Ser. A. **16**, 267–283 (1952).

A group $\mathfrak{G}$ of transformations $T(\alpha)$ of form $T(\alpha)$: $'\chi^r=\varphi^r(x,\alpha)=a_\mu{}^r(\alpha)x^\mu+$higher powers is majorized if there exist regular functions $\Phi^r(x)$ such that $\varphi^r(x,\alpha)\ll\Phi^r(x)$ for all α. This concept is applied to groups of transformations, to the Schröder functional equations, and to differential equations. For the case that $A=\|a_\mu{}^r\|$ has eigenvalues $\{\lambda_i\}$ all less (or all greater) than unity in magnitude, the Schröder equations have been solved (see fourth preceding review). It is here assumed that $|\lambda_i|=1$ for all i. The Schröder equations $f^r[\varphi(x)]=\lambda_r f^r(x)$ for a transformation T have regular solutions $f^r(x)=x^r+$higher powers if and only if $\{T^k\}$ $(k=0, 1, \cdots)$ is majorized.

Consider the differential equation (*) $\chi f\equiv\xi^\mu\partial f/\partial x^\mu=0$, where $\xi^\mu=c_\rho{}^\mu x^\rho+c^\mu{}_{\rho_1\rho_2}x^{\rho_1}x^{\rho_2}+\cdots$, and suppose the eigenvalues of $\|c_\rho{}^\mu\|$ lie on a line through the origin. Let $\mathfrak{G}$ be the one-parameter group with operator functions ξ^μ. Then $\mathfrak{G}$ is majorized if and only if the origin is stable for the differential equations $d'\chi^r/dt=\xi^r('\chi)$. Other results are obtained under the assumption that the eigenvalues of $\|c_\rho{}^\mu\|$ are pure imaginary, with absolute values that are mutually commensurable.

I. M. Sheffer (State College, Pa.).

Functional Analysis, Ergodic Theory

Altman, M. The Fredholm theory of linear equations in locally convex linear topological spaces. Bull. Acad. Polon. Sci. Cl. III. **2**, 267–269 (1954).

The author announces that the Fredholm theory of linear equations in Banach spaces developed by T. Leżański [Studia Math. **13**, 244–276 (1953); these Rev. **15**, 535] can be carried through in locally convex linear topological spaces (l.c.t.s.). The hypotheses parallel, in an obvious way, those of Leżański. It is assumed that T is a completely continuous transformation on the l.c.t.s. X to X in the sense that it maps some vicinity $\mathfrak{U}$ of 0 into a compact set [see Altman, ibid. **13**, 194–207 (1953); these Rev. **15**, 436]. The Fredholm determinants are then defined relative to the adjoint transformation $\hat{T}$ of T on Ξ (the space of additive functions bounded on $\mathfrak{U}$) to Ξ.

T. H. Hildebrandt.

Audin, Maurice. Sur certaines singularités des transformations linéaires bornées. C. R. Acad. Sci. Paris **238**, 2221–2222 (1954).

Let A be a bounded linear transformation of a Banach space E onto itself. Let $N(\lambda)$ be the set of null-elements of A [for definition see the review of Gol'dman and Kračkovskiĭ, Doklady Akad. Nauk (N.S.) **86**, 15–17 (1952); these Rev. **14**, 478]. The author calls λ a Riesz point if $N(\lambda)$ is finite-dimensional. He announces the following results. Every Riesz point which is an eigenvalue is isolated (which seems to the reviewer to be contained in the cited paper). If λ_0 is such a point, then $A=A_0+L_0$, where L_0 has only the eigenvalue λ_0 and A_0 has all eigenvalues of A except λ_0 [this result seems to be included in a paper of Kračkovskiĭ, same Doklady (N.S.) **88**, 201–204 (1953); these Rev. **14**, 1095]. Every Riesz point belongs to the Fredholm region. The Schmidt radius coincides with the Fredholm radius [conjectured by the author in C. R. Acad. Sci. Paris **237**, 511–512 (1953); these Rev. **15**, 233].

J. V. Wehausen (Providence, R. I.).

Dunford, Nelson. Spectral operators. Pacific J. Math. **4**, 321–354 (1954).

This is a continuation of the author's previously reported research on resolutions of the identity for a bounded linear operator in a complex Banach space X [same J. **2**, 559–614 (1952); these Rev. **14**, 479]. This paper and the five which are reviewed immediately hereafter are all closely related. Let $B(X)$ be the Banach algebra of bounded linear operators in X. If $\mathfrak{B}$ is the Boolean algebra of all Borel subsets σ of the complex plane p, a function E on $\mathfrak{B}$ into $B(X)$ is called a spectral measure if $E(\sigma_1\cap\sigma_2)=E(\sigma_1)E(\sigma_2)$, $E(\sigma_1\cup\sigma_2)=E(\sigma_1)+E(\sigma_2)-E(\sigma_1)E(\sigma_2)$, $E(\phi)=0$, $E(p)=I$, $E(p-\sigma)=I-E(\sigma)$, and $\sup_\sigma\|E(\sigma)\|<\infty$. Suppose that Γ is a total subspace in X^* and that E is a spectral measure such that $x^*E(\sigma)x$ is countably additive on $\mathfrak{B}$ for each $x\in X$ and each $x^*\in\Gamma$. Suppose that $T\in B(X)$ is such that it commutes with $E(\sigma)$ for every σ and such that the spectrum of T, as an operator on the range of $E(\sigma)$, lies in the closure of σ. Then T is said to be a spectral operator of class (Γ), and the spectral measure E is called a resolution of the identity for T; T is said to be spectral if it is spectral of class (Γ) for some Γ. A spectral operator has a unique resolution of the identity. It is moreover true that $E(\sigma(T))=I$ ($\sigma(T)$ the spectrum of T) and that if $A\in B(X)$ and $AT=AT$, then A commutes with $E(\sigma)$ for every σ. These results are obtained by considering the maximal analytic continuation of $(\lambda I-T)^{-1}x$. The fact that T is spectral insures that this extension is single-valued. If $\sigma(x)$ is the complement of the domain of existence of this maximal analytic continuation, we have the fundamental result that the range of $E(e)$ (e any Borel set) is the set of all x such that $\sigma(x)\subset e$.

The next part of the paper uses integrals $\int f(\lambda)E(d\lambda)$ of Riemann type in the uniform operator topology (integration over $\sigma(T)$), where f is a scalar function continuous on $\sigma(T)$. A spectral operator T is said to be of scalar type if $T=\int\lambda E(d\lambda)$, where E is the resolution of the identity for T. The central theorem of the whole paper is that T is a spectral operator of class (Γ) if and only if $T=S+N$, where S is a spectral operator of class (Γ) of scalar type and N is a generalized nilpotent operator which commutes with S. This decomposition is unique; T and S have the same spectrum and the same resolution of the identity. If f is a scalar function which is single-valued and analytic on a neighborhood of $\sigma(T)$, the operator $f(T)$ which is defined in a well-known manner by a contour integral is also expressible in the form

$$f(T)=\sum_{n=0}^{\infty}\frac{N^n}{n!}\int f^{(n)}(\lambda)E(d\lambda).$$

This series breaks off with $n=m$ for every f if and only if $N^{m+1}=0$. For $m=0$ T is of scalar type. The key to these developments is the theory of maximal ideals in Banach algebras. Operators of finite type in Hilbert space are characterized by a growth condition on $(\lambda I-T)^{-1}$ as an operator in the ranges of the projections $E(\sigma)$.

The last part of the paper deals with decomposition and representation theorems for commutative algebras of spectral operators. Sample theorem: Let $\mathfrak{A}$ be the (uniformly closed) algebra generated by a family of commuting spectral operators together with their resolutions of the identity. Suppose the Boolean algebra of the projections determined by these resolutions of the identity is bounded (i.e. that the norms of all the projections are uniformly bounded). Let $\mathfrak{R}$ be the radical in $\mathfrak{A}$. Then $\mathfrak{A}=\mathfrak{A}_1\oplus\mathfrak{R}$, where $\mathfrak{A}_1$ is an algebra which is equivalent to $C(\mathfrak{M})$, $\mathfrak{M}$ being the space of maximal ideals in $\mathfrak{A}$.

As a special characterization of a class of spectral operators, there is the following theorem: If X is reflexive and T is compact (i.e. completely continuous), T is spectral if and only if there is a constant M such that $\|\int_C(\lambda I-T)^{-1}d\lambda\|\leq M$ for every admissible contour in the resolvent set of T. *A. E. Taylor* (Geneva).

Wermer, John. Commuting spectral measures on Hilbert space. Pacific J. Math. 4, 355–361 (1954).

Let H be a Hilbert space. Definitions of spectral measures and spectral operators are as given by Dunford in the paper described in the immediately preceding review. The following two theorems are proved: (1) If E and F are spectral measures in H which commute (i.e. $E(\sigma_1)F(\sigma_2)=F(\sigma_2)E(\sigma_1)$ for every σ_1, σ_2), then there exists a bounded linear A with bounded inverse A^{-1} (defined on all of H) such that $A^{-1}E(\sigma_1)A$ and $A^{-1}F(\sigma_2)A$ are self-adjoint for every σ_1, σ_2. (2) If T_1 and T_2 are commuting spectral operators in H, then T_1+T_2 and T_1T_2 are also spectral. Theorem (1) is an extension of earlier results of Lorch [Bull. Amer. Math. Soc. 45, 564–569 (1939); these Rev. 1, 58] and Mackey (unpublished, given in lecture notes at Harvard). Theorem (2) follows from (1) with the aid of Dunford's decomposition of a spectral operator as a sum of an operator of scalar type and a generalized nilpotent operator (see the preceding review). The crucial step in the proof of (1) is the demonstration that: (3) $\sup\|\sum_{i=1}^{n}E(\sigma_i)F(\eta_i)\|<\infty$, where the supremum is taken over all finite systems $\sigma_1,\cdots,\sigma_n$ and $\eta_1,\cdots,\eta_n$ of Borel sets with $\sigma_1,\cdots,\sigma_n$ disjoint. The fact that H is a Hilbert space plays a vital role in these theorems, as is made clear by the counterexamples in the paper of Kakutani described in the following review.

A. E. Taylor (Geneva).

Kakutani, Shizuo. An example concerning uniform boundedness of spectral measures. Pacific J. Math. 4, 363–372 (1954).

The author demonstrates by counterexamples that the results (2) and (3) in Wermer's paper [cf. the preceding review] no longer hold true if the Hilbert space H is replaced by a suitable nonreflexive Banach space X. The status of these propositions remains undetermined for reflexive Banach spaces. In the counterexamples the space X is a cross-space $C(S)\otimes C(S)$, where S is a Cantor set of real numbers and $C(S)$ is the space of continuous functions on S.

A. E. Taylor (Geneva).

Bade, William G. Unbounded spectral operators. Pacific J. Math. 4, 373–392 (1954).

In this paper the author considers an extension of Dunford's theory of bounded spectral operators (see the review of the paper by Dunford reviewed third above) to the case of unbounded operators. In the present paper a spectral measure E is called a resolution of the identity (without reference to an operator) if $E(e)x$ is countably additive on $\mathfrak{B}$ (the Boolean algebra of all Borel sets in the plane) for each $x\in X$. An operator T with domain $\mathfrak{D}(T)$ and range in X is called spectral if it is closed and if there exists a resolution of the identity E such that: (1) $\mathfrak{D}(T)$ contains the subspace X_0 which is the union of all the ranges of $E(e)$ for all bounded e (the subspace X_0 is necessarily dense in X); (2) $E(e)\mathfrak{D}(T)\subset\mathfrak{D}(T)$ and T permutes with $E(e)$ on $\mathfrak{D}(T)$; (3) $\bar{e}$ contains the spectrum of the restriction of T to the range of $E(e)$. The uniqueness of E for a given T follows from the corresponding fact for bounded operators. T is a bounded operator on the range of $E(e)$ if e is bounded, so that Dunford's theory is applicable to the operator so restricted.

For each resolution of the identity E let

$$Sx=\lim_{n\to\infty}\int_{e_n}\lambda E(d\lambda)x,$$

with $\mathfrak{D}(S)$ the set of x for which the limit exists, where $e_n=\{\lambda\,|\,|\lambda|\leq n\}$. Any other increasing sequences of bounded Borel sets $\{e_n\}$ with $E(\bigcup_{n=1}^{\infty}e_n)=I$ would give the same limit. S is called the scalar operator associated with E, or with T, if E is the resolution of the identity of T. The operator S is spectral, and has the same resolution of the identity and the same spectrum as T. There is an operational calculus for S, analogous to that for unbounded normal operators in Hilbert space. If T is spectral, $E(\sigma(T))=I$, and if $\{e_n\}$ is an increasing sequence of bounded Borel sets such that $E(\bigcup_{n=1}^{\infty}e_n)=I$, then $\sigma(T)$ is the closure of the union of all the spectra of T restricted to the range of $E(e_n)$, $n=1, 2, \cdots$. With E and T given, S is defined, and then N is defined by $N=T-S$. It may be necessary to form a closed extension to define N suitably. It is shown by examples that N may be unbounded and also that it may be bounded but not a generalized nilpotent. However, if one starts with a scalar spectral operator S and a bounded operator N which commutes with the resolution of the identity for S and which is a generalized nilpotent on the range of $E(e)$ for each bounded e, then $T=S+N$ is a spectral operator whose resolution of the identity is the same as that of S. For any spectral operator T an operational calculus is constructed, starting from the formula

$$f(T)=\lim_{n\to\infty}\sum_{i=0}^{\infty}\frac{N^i}{i!}\int_{e_n}f^{(i)}(\lambda)E(d\lambda)x,$$

where f is one of a certain class of analytic functions and $\{e_n\}$ is a suitable increasing sequence of bounded sets. The operator $f(T)$ may not be spectral, but will be so if f is analytic on $\sigma(T)$ except for a finite set θ of poles for which $E(\theta)=0$, and if f is analytic or has a pole at ∞. Then the spectrum of $f(T)$ is the closure of $f(\sigma(T))$ and its resolution of the identity is defined by $E(f^{-1}(e))$.

An alternative to the definition as a criterion for a spectral operator is the following: A closed operator T with nonempty resolvent set is a spectral operator if and only if for some λ_0 non-$\in\sigma(T)$, $A=(\lambda_0 I-T)^{-1}$ is a bounded spectral operator (in Dunford's sense, of class X^*) for which $F((0))=0$, where F is the resolution of the identity for A and (0) is the set consisting of the single point $\lambda=0$ in the plane. This statement should replace the second sentence in the corollary on page 390, which is erroneous as printed. (The changed wording was communicated by the author.) It should be mentioned that, although the author does not explicitly point out the fact, his definition of an unbounded spectral operator imposes a limitation on the operator which amounts to requiring $F((0))=0$ as in the previous state-

ment. If this restriction were to be dropped, the theory would have to be modified to allow for a definition of $E(\infty)$, since ∞ must in a certain sense be counted as belonging to the spectrum of every unbounded operator. The operators here considered would have $E(\infty)=0$. In the work of J. Schwartz on operators with compact resolvents (see the second following review), his definition of an unbounded spectral operator makes allowance for the "spectral measure of ∞". Schwartz's theory does not apply to as broad a class of operators as does Bade's theory, however.

A. E. Taylor (Geneva).

Bade, William G. Weak and strong limits of spectral operators. Pacific J. Math. 4, 393–413 (1954).

The definitions of spectral operators of scalar type and of resolutions of the identity are as in the work of Dunford (see the fourth preceding review). Let V be a closed set in the complex plane, and let C be the Banach space (with the usual sup-norm) of continuous functions on V, vanishing at ∞ if V is unbounded. If V is nowhere dense in the plane and if the subset of C, composed of all rational functions which belong to C, is dense in C, then V is called an R-set. Alternative sufficient conditions for this are that V be of plane measure zero or that it not separate the plane. Let $\{T_\alpha\}$ be a net of spectral operators of scalar type, with corresponding resolutions of the identity $\{E_\alpha\}$ such that (1) the norms $\|E_\alpha(e)\|$ are uniformly bounded (all α, all Borel sets e) and (2) all the spectra $\sigma(T_\alpha)$ lie in some fixed R-set V. Finally, suppose that $T_\alpha x \to Tx$ (Moore-Smith convergence) for each $x \in X$. Then T^* is a spectral operator in X^* of scalar type and class (X). If X is reflexive, T itself is a scalar-type spectral operator of class (X^*). If T is of this latter type (whether X is reflexive or not), with resolution of the identity E, and if h is a bounded Borel measurable function on V whose set K of discontinuities is such that $E(K)=0$, then $h(T_\alpha)x \to h(T)x$ for each x. This generalizes a theorem of Kaplansky [same J. 1, 227–232 (1951); these Rev. 14, 291], and includes as a special case a theorem of Rellich [Math. Ann. 113, 677–685 (1937)] about convergence of resolutions of the identity of self-adjoint operators in Hilbert space.

The foregoing results are applied in investigating algebras generated by bounded Boolean algebras of projections in a reflexive space. A family $\mathfrak{B}$ of commuting projections, among them 0 and I, is called a Boolean algebra if it obeys the proper axioms with the operations $E_1 \vee E_2 = E_1+E_2-E_1E_2$, $E_1 \wedge E_2 = E_1E_2$. It is called bounded if the norms $\|E\|$ are uniformly bounded. It is called complete if it contains $\bigvee_\alpha E_\alpha$ and $\bigwedge_\alpha E_\alpha$ (defined in a natural way which we do not take the space to describe) for every subset $\{E_\alpha\}$ in $\mathfrak{B}$. Dunford [Bol. Soc. Mat. Mexicana 3, 1–12 (1946); these Rev. 9, 42] has discussed strong closure and completeness and shown that these concepts are equivalent for $\mathfrak{B}$ if it is bounded and X is reflexive. The present author gives a new proof of Dunford's result. With these same hypotheses on X and $\mathfrak{B}$, let $\mathfrak{B}'$ be the strong closure of $\mathfrak{B}$ and $\mathfrak{A}$ the weakly closed (or equivalently, strongly closed) algebra generated by $\mathfrak{B}$. It is shown that $\mathfrak{A}$ is composed of spectral operators of scalar type whose resolutions of the identity have all their projections in $\mathfrak{B}'$. Moreover, $\mathfrak{A}$ is identical with the uniformly closed algebra generated by $\mathfrak{B}'$. These results are used to generalize the characterizations given by von Neumann [Math. Ann. 102, 370–427 (1929)] and Segal [Mem. Amer. Math. Soc. no. 9 (1951); these Rev. 13, 472] of the weakly closed algebra generated by I and a self-adjoint operator T in Hilbert space. In the present paper the Hilbert space is replaced by a reflexive space, and T is any spectral, scalar-type operator with real spectrum. Another application of the results of the paper is to show that if $\mathfrak{A}$ is separable in the weak topology, then $\mathfrak{B}$ is separable in the strong topology. The converse is also true, and both conditions are equivalent to the condition that $\mathfrak{A}$ be the weakly closed algebra generated by I and a single element of $\mathfrak{A}$.

A. E. Taylor (Geneva).

Schwartz, J. Perturbations of spectral operators, and applications. I. Bounded perturbations. Pacific J. Math. 4, 415–458 (1954).

In this paper the author develops an abstract theory which can be applied to yield results concerning eigenvalue expansions for non-selfadjoint boundary-value problems in differential equations. The paper utilizes Dunford's theory of bounded spectral operators (see the fifth review before this one) and the reviewer's spectral theory for unbounded closed operators [Acta Math. 84, 189–224 (1951); these Rev. 12, 717]. Let X be a reflexive complex Banach space, and let T be a closed, densely defined operator whose resolvent set $\rho(T)$ is not empty, such that for some $\lambda \in \rho(T)$ (and, as it turns out, for all such λ), $(\lambda I-T)^{-1}$ is compact (=completely continuous). Such a T is called regular in this paper (surely a less hackneyed word had better be used if there is to be a standard terminology). If X is infinite-dimensional, T must be unbounded. The spectrum of T is an at most denumerably infinite set $\{\lambda_n\}$ with no finite point of accumulation. Each λ_n is a pole of $(\lambda I-T)^{-1}$ of some order $\nu(\lambda_n)$, and is in the point-spectrum. Associated with λ_n is a projection $E(\lambda_n)$ whose range is the null space of $(\lambda_n I-T)^{\nu(\lambda_n)}$. Projections corresponding to different points are orthogonal. T is called spectral if $(\lambda I-T)^{-1}$ is spectral in the sense of Dunford for some $\lambda \in \rho(T)$; it is then spectral for all such λ. By a theorem of Dunford (stated at the end of the foregoing review cited above) it turns out that T is spectral if and only if the projections $E(\lambda_n)$ generate a bounded Boolean algebra as defined in the preceding review of a paper by Bade. The reflexivity of T enters at this point.

If T is regular, but not necessarily spectral, $\mathrm{sp}(T)$ denotes the smallest closed linear manifold containing the ranges of all the $E(\lambda_n)$, and $S_\infty(T)$ denotes the intersection of the null manifolds of all the $E(\lambda_n)$. It turns out that $S_\infty(T)$ is either infinite-dimensional or is (0) (consisting only of 0). If T^* is the Banach adjoint of T, regularity of T^* is equivalent to that of T, and if both are regular and one is spectral, so is the other. For regular T, $\mathrm{sp}(T)=(S_\infty(T^*))^\perp$. For a regular and spectral T, the series $\sum_n E(\lambda_n)$ converges in the strong topology of operators and defines a projection F whose range is $\mathrm{sp}(T)$. In this case $F=I$ is equivalent to $\mathrm{sp}(T)=X$ and to $S_\infty(T)=(0)$, but if T is not spectral it can happen that $\mathrm{sp}(T)\neq X$ and $S_\infty(T)=(0)$. Suppose T is regular and spectral, with $S_\infty(T)=(0)$. Let d_n be the distance from λ_n to the rest of the spectrum. Suppose that λ_n is a simple pole of the resolvent for almost all values of n (i.e., for all but a finite number). Let B be any bounded operator. Then: (a) If $d_n \to \infty$, $T+B$ is regular. (b) If $\liminf d_n > 0$, $T+B$ is regular if $\|B\|$ is sufficiently small or if B is compact. Under suitable conditions, which we do not take space to state here, it may be concluded also that $S_\infty(T+B)=(0)$. Other results: If we further assume that for almost all n the range of $E(\lambda_n)$ is 1-dimensional (which implies the restriction to simple poles), then $\sum d_n^{-1} < \infty$ implies that $T+B$ is spectral, with $S_\infty(T+B)=(0)$. The same conclusions follow from $\sum d_n^{-2} < \infty$ if X is a Hilbert space.

In applying these results to differential operators and other unbounded operators in problems of classical analysis, the idea is to choose T so that it is regular and spectral, with $S_\infty(T)=(0)$, and so that the operator to be investigated has the form $T+B$, with bounded B. As an example, let X be the space $L^2(0,1)$, and let A^2 be the class of functions f continuous on $[0,1]$, with f' absolutely continuous and $f''\in L^2$. The space A^2 is a Banach space with norm

$$\left\{\int_0^1 |f''(x)|^2dx\right\}^{1/2}+\max_{0\le x\le 1}\max\{|f(x)|,|f'(x)|\}.$$

Let T be the operator in L^2 defined by the formal differential operator d^2/dx^2, with domain of T the subset of A^2 satisfying linear boundary conditions of the standard sort. Then T is a regular spectral operator satisfying the aforementioned conditions with $\sum d_n^{-2}<\infty$, so $T+B$ is spectral and $S_\infty(T+B)=(0)$ for any bounded B. In particular, $T+B$ might be defined by $d^2/dx^2+q(x)$, where q is bounded and measurable. In this case, if $\{E(\lambda_n)\}$ are the projections determined by the spectrum of $T+B$, the series $\sum E(\lambda_n)f$ converges to f in the topology of L^2 for each $f\in L^2$, and converges to f in the topology of A^2 for each f in the domain of T. Some applications to singular ordinary differential equations and to elliptic partial differential equations are also discussed. *A. E. Taylor* (Geneva).

Leader, Solomon. On the infinitesimal generator of a semigroup of positive transformations with local character condition. Proc. Amer. Math. Soc. **5**, 401–406 (1954).

The paper gives an extension of the result of W. Feller in Ann. Soc. Polon. Math. **25**, 85–94 (1953) [these Rev. **14**, 1094]. Let $\{T_t\}$, $t>0$, be a one-parameter semi-group of linear operators on the space C of bounded continuous functions on a closed interval of the real line satisfying the conditions: (i) $f(x)\ge 0$ for all x implies $T_tf(x)\ge 0$ for all x and t; (ii) if $f(x)$ vanishes throughout a vicinity of x, then $T_tf(x)=o(t)$; (iii) $T_t\cdot 1=1$. Let, moreover, Ω be the infinitesimal generator of T_t, viz.

$$\Omega f(x)=\text{uniform}\lim_{t\to+0} t^{-1}(T_tf(x)-f(x))$$

in case the right-hand limit exists (we denote the domain of this linear operator Ω by D). Then the above three conditions imply the following lemma (due to W. Feller): If $f\in D$ and if $f(x)$ has a local minimum (local maximum) at $x=x_0$, then $\Omega f(x_0)\ge 0$ ($\Omega f(x_0)\le 0$). By making use of this lemma, the author proves that, at a fixed point x, one of the four cases below must hold. (I) $\Omega f(x)=0$ for all $f\in D$. (II) There exists $v\in D$ with $v\ge 0$ near x and $\Omega v(x)>0$ such that $\Omega f(x)=\rho\Omega v(x)$, $\rho=\rho(f,x)$. (III) There exists $u\in D$ with u strictly increasing near x and $\Omega u(x)\ne 0$ such that $\Omega f(x)=\sigma\Omega u(x)$, $\sigma=\sigma(f,x)$. (IV) There exist both v and u of the properties respectively given in (II) and (III) such that $\Omega f(x)=\sigma\Omega u(x)+\rho\Omega v(x)$. Hence, in this case, Ω will correspond to a second-order differential operator at x whenever it is possible to choose u and v in such a way that $u(x)=v(x)=0$ and $u=v+o(v)$ in the vicinity of x.

K. Yosida (Osaka).

Kampé de Fériet, Joseph. Transformations de Reynolds opérant dans un ensemble de fonctions mesurables non négatives. C. R. Acad. Sci. Paris **239**, 787–789 (1954).

The concept of a "Reynolds operator" T on a function space, introduced by the author [La Science Aérienne **3**, 9–34 (1934); **4**, 12–52 (1935)], has recently been analyzed by S.-T. C. Moy [Pacific J. Math. **4**, 47–63 (1954); these Rev. **15**, 722] for the case of the function space M of measurable functions. In the present note, the author gives three necessary and sufficient conditions on the transforms $T(c_E)$ of characteristic functions c_E, for T to define a Reynolds operator on M. *G. Birkhoff.*

Dubreil-Jacotin, Marie-Louise. Propriétés générales des transformations de Reynolds. C. R. Acad. Sci. Paris **239**, 856–858 (1954).

The author continues her researches concerning Reynolds operators [same C. R. **236**, 1136–1138, 1950–1951 (1953); these Rev. **14**, 839; **15**, 194], considering Reynolds operators T on spaces of functions assuming finitely many different values. The values are supposed in a lattice-ordered integral domain. Typical result: if $Tf=f$, then each class of the partition $\pi(f)$ is the union of disjoint sets A and B whose characteristic functions satisfy $Tc_A=c_A$ and $Tc_B=0$.

G. Birkhoff (Cambridge, Mass.).

Arbault, Jean. Nouvelles propriétés des transformations de Reynolds. C. R. Acad. Sci. Paris **239**, 858–860 (1954).

In case the values of the functions treated are in a simply ordered integral domain, the results of the preceding note can be slightly extended. *G. Birkhoff.*

Régnier, André. Sur les opérateurs de Koopman. C. R. Acad. Sci. Paris **238**, 1857–1858 (1954).

Apparently elementary comments on the relation connecting homomorphisms between Boolean algebras of sets and linear transformations between some corresponding function spaces; the cryptographic style customary in such notes has been carried in this one to the point of rendering it almost unintelligible. *P. R. Halmos* (Chicago, Ill.).

Williamson, J. H. On topologising the field $C(t)$. Proc. Amer. Math. Soc. **5**, 729–734 (1954).

Arens [Bull. Amer. Math. Soc. **53**, 623–630 (1947), Theorem 1; these Rev. **9**, 6] generalized the classical result of Mazur and Gelfand that the complex field C is the only complete normed field over C. The author improves Arens' result, and derives from this improvement that C is the only complex topological division algebra of type V [in the sense of Kaplansky, ibid. **54**, 809–826 (1948); these Rev. **10**, 179].

He then turns to the problem, posed by Kaplansky [loc. cit.], of topologizing the field $C(t)$ of all rational functions in the indeterminate t over the field C. First, $C(t)$ is given a metrisable topology in which addition, multiplication, and inversion are continuous. Next, $C(t)$ is given a locally convex metrisable topology in which addition and multiplication are continuous. If it is further required that $C(t)$ be a complex topological linear space, the topology (whether locally convex or not) cannot be too coarse or too fine. In the latter case, it is shown that in the finest (locally convex) topology on $C(t)$ compatible with its linear space structure, multiplication is not continuous (although it is continuous in each factor). *M. Henriksen* (Lafayette, Ind.).

Guy, Roland. Sur l'existence des solutions de systèmes finis d'équations fonctionnelles non linéaires. C. R. Acad. Sci. Paris **239**, 229–231 (1954).

Consider a system of functional equations

$$U_i=U_i^0+\lambda\sum F_{ij}(U_j)\quad(i,j=1,\cdots,n)$$

where U_i, U_i^0 are elements of a complete unitary A-module

A over the complex field C with distance function d, and the F_{ij} are non-linear transformations defined in A such that domain $F_{ij}\supset$counter-domain F_{ij}. It is stated that if Lipschitz conditions of the form

$$d(F_{ij}(U')-F_{ij}(U''))\leqq|f_{ij}|d(U'-U'')$$

hold, then the series $U_i^{(0)}+\sum_k U_i^{(k)}-U_i^{(k-1)}$ where $U_i^{(0)}=U_i^0$, $U_i^{(n)}=\lambda\sum F_{ij}U_j^{(n-1)}$, converges to a unique solution of the given system provided $|\lambda|$ is sufficiently small. The author remarks that if A is a set of bounded linear transformations of a Banach space, the topology in the sense of d can be replaced by any of the standard ones which are appropriate.

P. A. Smith (New York, N. Y.).

Vorob'ëv, Yu. V. Operator orthogonal polynomials and approximate methods of determination of the spectrum of linear bounded operators. Uspehi Matem. Nauk (N.S.) **9**, no. 1(59), 83–90 (1954). (Russian)

Let $A=\int_m^M\lambda dE_\lambda$ be a bounded self-adjoint linear operator defined on a Hilbert space H. For an arbitrary $z_0\in H$ assume one knows the elements $z_0, z_1=Az_0, \cdots, z_{n-1}=A^{n-1}z_0$. Two problems are solved by use of operator orthogonal polynomials: I. To use the $z_0, \cdots, z_{n-1}$ to approximate the spectral function $dE_\lambda z_0$, without any restriction on the nature of the spectrum. II. To determine the weighted sum $u_0^{(n-1)}$ of $z_0, \cdots, z_{n-1}$ for which the Rayleigh quotient

$$\lambda_0^{(n-1)}=(Au_0^{(n-1)}, u_0^{(n-1)})/(u_0^{(n-1)}, u_0^{(n-1)})$$

is the largest, and so get a best approximation to the largest eigenvalue of A, under the assumption that A has a pure point spectrum.

Let $\bar z_n$ be the projection of $z_n=Az_{n-1}$ on the subspace spanned by $z_0, \cdots, z_{n-1}$. The author first solves what he calls the "restricted moment problem in Hilbert space": to find an operator B_n such that $z_k=B_n{}^k z_0$ $(0\leqq k\leqq n-1)$ and $\bar z_n=B_n{}^n z_0$. In the solution, polynomials $P_k(\lambda)$ are defined in terms of the Schwarz constants $s_{i+k}=(z_i, z_k)$, with the property that

$$\int_m^M P_j(\lambda)P_k(\lambda)d(E_\lambda z_0, z_0)=\delta_{jk}.$$

To solve problem I the author uses the spectral function $E_\lambda B_n z_0$ of the restricted moment problem, and proves that it converges strongly to $E_\lambda z_0$. In the solution of problem II it turns out that the maximal $\lambda_0^{(n-1)}$ is the largest zero of $P_n(\lambda)$.

The polynomials $P_k(\lambda)$ and the corresponding orthogonality properties of the $P_k(A)z_0$ were used by C. Lanczos [J. Research Nat. Bur. Standards **45**, 255–282 (1950); these Rev. **13**, 163] and Hestenes and Stiefel [ibid. **49**, 409–436 (1953); these Rev. **15**, 651].

G. E. Forsythe.

Putnam, C. R. Remarks on certain operators of quantum field theory. J. London Math. Soc. **29**, 350–354 (1954).

If A is a closed densely defined linear operator on a Hilbert space with the properties that AA^* and A^*A have the same domain and that AA^*-A^*A is extended by the identity, then A^*A has a purely discrete spectrum consisting of all non-negative integers, and the point spectrum of A consists of all complex numbers.

I. E. Segal.

Takeda, Zirô. Conjugate spaces of operator algebras. Proc. Japan Acad. **30**, 90–95 (1954).

Soit A une B^*-algèbre. Lemme: Toute forme linéaire continue sur A est une combinaison linéaire finie d'états (states) sur A. (On représente les éléments hermitiens de A comme des fonctions continues sur l'espace des états, et on utilise la décomposition des mesures en combinaisons linéaires de mesures positives). Si toute forme linéaire continue sur A est ultrafaiblement continue, le bidual de A s'identifie canoniquement à l'adhérence faible de A: on utilise le théorème de densité de Kaplansky [Pacific J. Math. **1**, 227–232 (1951); ces Rev. **14**, 291]. Dans le cas général, il existe un isomorphisme de A sur une *-algèbre qui possède la propriété précédente. Donc le bidual de A s'identifie à un anneau d'opérateurs. L'auteur donne aussi une caractérisation abstraite du sous-espace des formes linéaires ultrafortement continues dans le dual d'un anneau d'opérateurs.

J. Dixmier (Paris).

Takeda, Zirô. On the representations of operator algebras. Proc. Japan Acad. **30**, 299–304 (1954).

Soit A une B^*-algèbre. L'auteur caractérise les ensembles d'états sur A qui, dans une C^*-représentation convenable (ou dans une C^*-représentation donnée à l'avance), deviennent les états ultrafortement continus. Pour que A admette une W^*-représentation, il faut et il suffit qu'il existe un ensemble "rudimentaire" d'états sur A.

J. Dixmier.

Cotlar, M. On a theorem of Beurling and Kaplansky. Pacific J. Math. **4**, 459–465 (1954).

Let G be a locally compact Abelian group, $\hat G$ its character group, and $\hat f$ the Fourier transform on $\hat G$ of a function $f\in\mathfrak{L}_1(G)$. Using methods due to A. Beurling, Helson has proved [Ark. Mat. **1**, 497–502 (1952); these Rev. **14**, 246] that a closed ideal $\mathfrak{I}$ in $\mathfrak{L}_1(G)$ is the intersection of the maximal ideals containing it if the boundary in $\hat G$ of the set $S(\mathfrak{I})=\bigcap_{f\in\mathfrak{I}}\{\hat x;\hat f(\hat x)=0\}$ contains no non-void perfect set. The present paper proves this theorem as well as the special case, due to Beurling and Kaplansky, in which $S(\mathfrak{I})$ consists of a single point. The proof adapts a device used by Zygmund [Trigonometrical series, Warszawa-Lwów, 1935, p. 141], which is originally due to Wiener [Ann. of Math. (2) **33**, 1–100 (1932)].

E. Hewitt (Seattle, Wash.).

Mautner, F. I. Geodesic flows and unitary representations. Proc. Nat. Acad. Sci. U. S. A. **40**, 33–36 (1954).

The geodesic flow on a complete surface of constant negative curvature can be represented as a flow defined on a coset space G/Γ, where G is the group of all real 2×2 matrices of determinant 1 and Γ is a discrete subgroup of G. A brief operator-theoretic proof of the ergodicity of this flow, when the (unique invariant) measure of G/Γ is finite is sketched and thus the ergodicity of the geodesic flow on a complete surface of constant negative curvature of finite area is shown. The method is simpler than that of Gelfand and Fomin [Doklady Akad. Nauk SSSR (N.S.) **76**, 771–774 (1951); Uspehi Matem. Nauk (N.S.) **7**, no. 1(47), 118–137 (1952); these Rev. **13**, 473; **14**, 660], but yields less information. It is stated that the method can be generalized in various ways and, in particular, can be applied to the Riemannian manifolds obtained by identification modulo a properly discontinuous group of motion from locally symmetric Riemannian manifolds of negative curvature. An additional theorem concerning the ergodicity of double cosets is derived when G is a locally compact topological group and K is a closed subgroup of G.

G. A. Hedlund.

Theory of Probability

***Savage, Leonard J. The foundations of statistics.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1954. xv+294 pp. $6.00.

L'auteur expose comment on peut fonder la théorie générale des probabilités et la statistique selon ses vues personnelles; celles-ci se rattachent aux conceptions subjectivistes, en particulier à celles de Finetti, mais leur originalité est de partir systématiquement d'une théorie du "comportement." Dans les circonstances comportant de l'incertitude une personne a, en premier comme en dernier lieu, à décider parmi plusieurs actes possible celui qu'elle choisit d'exécuter: tout doit donc découler de cette inéluctable obligation de décider. Autant que possible l'exposé élimine les complications purement mathématiques au profit du développement des idées, que voici résumé (évidemment trés grossiérement):

Appelons: univers: le système, auquel se rapporte la décision, et dont l'état s est aléatoirement pris parmi l'ensemble S des états possibles; événement: une sous-ensemble quelconque de S; conséquence: n'importe quoi pouvant arriver à la personne, selon la décision qu'elle prend et l'état réel de l'univers; F: l'ensemble de toutes les conséquences possibles; acte: une application de S sans F, c'est à dire une fonction attachant, à chaque état possible s de l'univers, une conséquence $f=f(s)$.

L'auteur postule que la personne classe par "ordre de préférence" les divers actes possibles et que ce classement constitue, dans l'ensemble des actes, une relation d'ordre simple (avec transitivité); il montre que ce postulat, convenablement complété de quelques autres moins typiques, implique l'existence entre les événements d'une relation d'ordre simple dite "probabilité qualitative," qui, dans certains cas pour lesquels il donne des critères suffisants, s'exprime de façon unique par une mesure de probabilité (probabilité quantitative) de type classique. Ces points font l'objet des trois premiers chapitres, le troisiéme comportant en outre l'introduction des probabilités conditionnelles, et l'examen des conditions dans lesquelles une répétition suffisante de l'expérience permet d'approcher la certitude; tandis que le chapitre IV examine critiquement la conception de l'auteur par rapport à d'autres.

Au chapitre V, l'auteur montre que l'ordre de préférence entre les actes peut être exprimé (de façon unique, à une transformation linéaire près) par une fonction numérique de $f \in F$, dite: utilité (la démonstration est essentiellement empruntée à von Neumann et Morgenstern); puis il fait un exposé historique et critique sur cette notion d'utilité.

Le reste de l'ouvrage consiste à utiliser les notions ci-dessus pour examiner, critiquer, repenser, etc. les problèmes et les méthodes fondamentaux de la théorie des jeux et de la statistique. Il serait difficile ici de résumer cette partie de façon intéressante. *R. Fortet* (Paris).

Clarke, R. D. The concept of probability. J. Inst. Actuar. 80, 1–12; discussion, 13–31 (1954).

Le texte reproduit une conférence prononcée par l'auteur, à l'issue de laquelle se developpa une discussion entre lui et divers auditeurs, discussion qui est rapportée à la suite du texte. En se referrant fréquemment aux problèmes professionnels de l'actuariat, l'auteur discute les deux conceptions de la probabilité, comme rattachée à la fréquence, ou comme exprimant un degré de croyance; il soutient qu'on ne peut se passer de la seconde, et qu'on peut lui donner une acception convenable pour que, loin de s'opposer à la première, elle l'englobe comme cas particulier.
R. Fortet (Paris).

de Finetti, Bruno. Gli eventi equivalenti e il caso degenere. Giorn. Ist. Ital. Attuari 15, 40–64 (1952).

The author analyzes a sequence of equivalent events, that is, a sequence of Bernoulli trials in which the common success probability p is itself a random variable. The most emphasis is laid on the degenerate case in which the distribution of p is confined to the points 0, 1, and on cases near the degenerate case. *J. L. Doob* (Urbana, Ill.).

Daboni, Luciano. Aspetti di una interpretazione geometrica per le probabilità di eventi equivalenti. Rivista Mat. Univ. Parma 4, 145–165 (1953).

Daboni, Luciano. Considerazioni geometriche sulla condizione di equivalenza per una classe di eventi. Giorn. Ist. Ital. Attuari 16 (1953), 58–65 (1954).

The author considers a sequence of equivalent events, using a graphical representation due to de Finetti (see the preceding review) and, in particular cases, correlates to some extent the graphical representation with the distribution of success probability p. *J. L. Doob* (Urbana, Ill.).

de Finetti, Bruno. Trasformazioni di numeri aleatori atte a far coincidere distribuzioni diverse. Giorn. Ist. Ital. Attuari 16 (1953), 51–57 (1954).

Soient X et Y deux variables aléatoires, de fonctions de répartition continues différentes F et G; soit H une fonction de répartition continue arbitraire; l'auteur montre qu'on peut trouver une fonction (continue) ϕ et une seule telle que $X'=\phi(X)$ et $Y'=\phi(Y)$ aient toutes deux H comme fonction de répartition. Il signale d'intéressantes applications ou interprétations de ce résultat. *R. Fortet.*

de Finetti, Bruno. Una legge riguardante l'estinzione nei processi di eliminazione. Giorn. Ist. Ital. Attuari 16 (1953), 94–99 (1954).

The probability that an individual is "eliminated" in any one of a (large) number of successive intervals is q. There are n such individuals who simultaneously become subject to elimination. It is shown that the probability that the last m survivors are all eliminated in one interval is $(p=1-q)$

$$P_m^{(n)}=q^m\sum_{h=0}^{\infty}\binom{n}{m}p^{hn}(1-p^h)^{n-m}\simeq q^m/m\,|\ln p|=P_m.$$

H. L. Seal (New York, N. Y.).

Ottaviani, Giuseppe. A proposito della legge di estinzione nei processi di eliminazione. Giorn. Ist. Ital. Attuari 16 (1953), 100–114 (1954).

The author shows that $P_m^{(n)}$ of the preceding review has no unique limit as $n\to\infty$; that P_m is a mean of the points of accumulation of $P_m^{(n)}$; and that easily calculated approximations can be found for the fluctuation of $P_m^{(n)}$ as $n\to\infty$.
H. L. Seal (New York, N. Y.).

Lévy, Paul. Trois théorèmes de calcul des probabilités. C. R. Acad. Sci. Paris 238, 2283–2286 (1954).

The first theorem concerning characteristic functions is a special case of a theorem of Bochner [Ann. of Math. (2) 48, 1014–1061 (1947), Theorem 10; these Rev. 9, 193]. The second concerns a stochastic area. The third concerns the force of attraction engendered by a Poissonian distribution of masses. *K. L. Chung* (Syracuse, N. Y.).

Nyquist, H., Rice, S. O., and Riordan, J. The distribution of random determinants. Quart. Appl. Math. **12**, 97–104 (1954).

Soit D_n un déterminant d'ordre n, dont les élements sont des variables aléatoires mutuellement independantes, de même distribution symétrique par rapport à 0, et d'écart moyen quadratique égal à 1. Les moments d'ordre impair de D_n sont nuls; les auteurs donnent la valeur du moment d'ordre 2 [déjà donnée par R. Fortet dans J. Research Nat. Bur. Standards **47**, 465–470 (1951); ces Rev. **13**, 852] et du moment d'ordre 4; la valeur du moment d'ordre 6 pour $n=1, 2, 3, 4$. Dans le las particulier où les éléments de D_n sont des variables aléatoires laplaciennes, ils donnent l'expression générale du moment d'ordre $2K$ [résultat qui, comme le signalent les auteurs, peut être obtenu autrement, cf. G. Forsythe et J. W. Tukey, Bull. Amer. Math. Soc. **58**, 502 (1952)], et de la densité de probabilité de D_n; celle-ci est particulièrement simple pour $n=1, 2, 3, 4$.

R. Fortet (Paris).

Robbins, Herbert. A remark on the joint distribution of cumulative sums. Ann. Math. Statistics **25**, 614–616 (1954).

Let X_k, $k=1, \cdots, n$, be independent random variables. Let $T_k=X_1+\cdots+X_k$,

$$G_k(t)=\Pr[T_k\leqq t],$$

and

$$G(t_1, \cdots, t_n)=\Pr[T_1\leqq t_1, \cdots, T_n\leqq t_n].$$

The author shows that $G(t_1, \cdots, t_n)\geqq\prod_1^n G_k(t_k)$.

J. Wolfowitz (Ithaca, N. Y.).

Stanoyevitch, Tchaslav. Sur une généralisation d'une inégalité de M. Kolmogoroff. C. R. Acad. Sci. Paris **239**, 854–856 (1954).

The author generalizes the well known inequality of Kolmogoroff.

J. Wolfowitz (Ithaca, N. Y.).

Barbot, Jacques. Etude mathématique généralisée du jeu de Nain Jaune. Bull. Trimest. Inst. Actuaires Français **64**, 95–348 (1953).

This is a thesis on a game called Grand Lindor, defined by the author, and including a number of well-known card games as special cases; for example, Pope Joan (Lindor or Nain Jaune), Loterie, Bog, Guimbarde and Poque. Grand Lindor itself is not necessarily a card game. There are c containers ($c\geqq 2$) containing stakes $u_{h,n}$ ($h=1, 2, \cdots, c$; $n=1, 2, 3, \cdots$). These stakes are random variables satisfying equations of the form $u_{h,n+1}=i_{h,n}u_{h,n}+m_{h,n+1}$, where $m_{h,n}$ is a non-probabilistic function of h and n, while $i_{h,n}$ is a random variable that takes the value i ($i=0, 1, 2, \cdots, l$) with probability $\theta_{h,n}^{(i)}$ ($\sum_{i=0}^{l}\theta_{h,n}^{(i)}=1$). The rules describing how the individual players are affected are irrelevant to the general theory, though they are used in order to tabulate the functions $m_{h,n}$ and $\theta_{h,n}^{(i)}$ for the special games. The general discussion is mainly concerned with the mathematical expectations of the stakes and with whether they have limiting distributions independent of their initial values $u_{h,1}$. The subject belongs largely to the theory of probabilities in chains.

I. J. Good (Cheltenham).

Bailey, Norman T. J. On queueing processes with bulk service. J. Roy. Statist. Soc. Ser. B. **16**, 80–87 (1954).

Bulk service consists of service in batches, a batch consisting of at most s customers. Customers arrive at random, with mean number of arrivals λ, are formed into a queue in order of arrival when the server is busy, and the batch is less than s when the queue is less than s at the time of commencing service. It is shown that the system is stable when $\rho=\lambda\bar{v}/s$ is less than unity, $\bar{v}$ being the average service time. [It may be noted that when the batches are always exactly s, the system is a single server system, which is stable for arrivals with "limited interaction," that is for independent inter-arrival times each with the same distribution, when $\rho<1$, where $\rho=\bar{v}/sa$, and a is the mean inter-arrival time.] The generating function for queue length (in statistical equilibrium) is given in a form closely resembling that for congestion probabilities of the s-server system with Poisson arrivals and constant service time, the same for each server (Crommelin). This is examined in detail for Erlangian (χ^2) distribution of the service time. Also the mean waiting time is determined for general service time. Finally, inequalities both for mean queue length and mean waiting time are developed to simplify the study of hospital out-patient department practices, the setting in which the problem arose. The author mentions possible applications in other fields such as "transport processes involving lifts or buses"; it may be interesting to notice that bulk service also appears in telephone switching.

J. Riordan (New York, N. Y.).

Domb, C. On multiple returns in the random-walk problem. Proc. Cambridge Philos. Soc. **50**, 586–591 (1954).

Generalization of Polya's random walk problem to sums of independent, identically distributed random variables with zero mean but no further moment was given by Chung and Fuchs [Mem. Amer. Math. Soc. no. **6** (1951); these Rev. **12**, 722], and a special case (symmetrical lattice distribution) of it was given by Foster and Good [Quart. J. Math., Oxford Ser. (2) **4**, 120–126 (1953); these Rev. **14**, 1101]. Here the author gives an asymptotic evaluation of the probability of return at the nth step for a bounded lattice distribution. General expansions of this kind have been given by Wintner, Esseen and others [see Esseen, Acta Math. **77**, 1–125 (1945); these Rev. **7**, 312].

K. L. Chung (Syracuse, N. Y.).

Yanoši [Janossy], L. Investigations on the theory of cascades. I. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **26**, 386–404 (1954). (Russian)

This is largely an expository treatment of cascade processes. A primary particle of a certain energy collides, loses some of its energy and creates secondary particles of the same or of different types. Probabilities corresponding to the various possibilities are introduced, the "catastrophic" disappearance of particles as well as the continuous loss of energy are also considered and kinetic equations are given.

E. Lukacs (Washington, D. C.).

Ramakrishnan, Alladi, and Srinivasan, S. K. Two simple stochastic models of cascade multiplication. Progress Theoret. Physics **11**, 595–603 (1954).

The authors consider a multiplicative process which has qualitative features similar to cosmic-ray cascade processes. Let $2w(q)dq$, $0<q<1$, be the probability per unit thickness of matter that a particle of energy E splits into two, one with energy between Eq and $E(q+dq)$, the other between $E(1-q)$ and $E(1-q-dq)$. The initially incident particle has energy E_0; it is desired to calculate the mean and mean square number of particles above an energy E_c at thickness t. Let $f_1(E, t)$ be the product-density of degree 1; that is, the expected number between energies E_1 and E_2 at thick-

ness t is $\int_{E_1}^{E_2} f(E, t)dE$. Then f_1 satisfies an integro-differential equation whose solution can be calculated using the Mellin transform. The second moment is treated similarly. The first moment is found explicitly for the case $w(q)=$constant. A case where $\int_0^1 w(q)dq=\infty$ is also treated, namely $w(q)=\frac{1}{2}(q^{-1}+(1-q)^{-1})$. [The precise nature of the underlying stochastic process in this case is not clear to the reviewer.] Numerical tables of first and second moments for the two cases are given. *T. E. Harris.*

Ramakrishnan, Alladi. Counters with random dead time. Philos. Mag. (7) **45**, 1050–1052 (1954).

Events occur at random in time according to a Poisson process with constant mean rate. Type I counters: every registered event is followed by a random dead time. Type II: every event is followed by a random dead time. There is no registration during dead times. Assuming that the dead times are independent random variables with a common distribution and that an initial event is registered at time 0, the author finds for the two types the distribution of the number of events registered in a given time.

T. E. Harris (Santa Monica, Calif.).

Takano, Kinsaku. On some limit theorems of probability distributions. Ann. Inst. Statist. Math., Tokyo **6**, 37–113 (1954).

In Part I Khintchine's uniqueness theorem for the class convergence of probability distributions is proved in a natural way by making use of inverses of distribution functions; its generalization to the multi-dimensional case is also proved; relations between different paired sequences of scaling constants and centering constants in limit problems of probability distributions are given; and the general method to determine scaling constants and centering constants is presented. Most of the results in Part I of this paper have been given in the writers previous papers: Ann. Inst. Statist. Math., Tokyo **3**, 7–15 (1951); **5**, 1–7, 41–58 (1953) [these Rev. **13**, 566; **15**, 329].

In Part II both an analytical derivation of P. Levy's canonical form of the infinitely divisible multi-dimensional probability distributions and a necessary and sufficient condition for the distributions of sums of asymptotically uniformly negligible independent multi-dimensional random variables to converge to a given infinitely divisible probability distribution are given. The logarithms of non-vanishing characteristic functions are treated rigorously.

In Part III various versions of the multi-dimensional central limit theorem on sums of independent random variables are studied.

The results in the last two parts are extensions of the known facts in the one-dimensional case to the multi-dimensional case. (Author's summary.) *J. Wolfowitz.*

Fortet, R., et Mourier, E. Résultats complémentaires sur les éléments aléatoires prenant leurs valeurs dans un espace de Banach. Bull. Sci. Math. (2) **78**, 14–30 (1954).

The authors consider Banach-space-valued random variables, proving various forms of the law of large numbers, for sums of mutually independent random variables, and correcting an earlier version [C. R. Acad. Sci. Paris **234**, 699–700 (1952); these Rev. **14**, 387]. If the range Banach space is a separable Hilbert space, and if Y is a Laplacian random variable, they evaluate the characteristic function of $\|y\|^2$, and prove that, if $Z_n=n^{-1/2}\sum_1^n Y_j$, where the Y_j's are mutually independent, with the common distribution of Y, which is supposed to have zero mean, then the distribution of $f(Z_n)$ approaches that of $f(Y)$, if f is a numerically valued function, uniformly continuous on the unit sphere.

J. L. Doob (Urbana, Ill.).

Segal, I. E. Abstract probability spaces and a theorem of Kolmogoroff. Amer. J. Math. **76**, 721–732 (1954).

A probability algebra is defined as a commutative algebra R over the real field, together with a linear functional E, having the properties: $E(a^2)\geq 0$, with equality only if $a=0$; to each a in R corresponds a constant μ such that $E(ab^2)\leq \mu E(b^2)$ for every b in R; R has a unit e, with $E(e)=1$. It is shown that a probability algebra is isomorphic (with algebraic operations and E value preserved) to a set of bounded random variables on a probability measure space, for which the E value is expectation. There is uniqueness here, in an appropriate sense, if the class of measurable sets is chosen properly. The analogue for probability algebras of the Kolmogorov theorem on the existence of a family of random variables with any preassigned joint distributions is proved. The proof is simpler than the usual one, because in this treatment no coordinate spaces are specified in advance. Possible simplifications of standard treatments of stochastic processes are mentioned, as is the extension of the theory to non-commutative probability algebras.

J. L. Doob (Urbana, Ill.).

Chung, K. L. Contributions to the theory of Markov chains. II. Trans. Amer. Math. Soc. **76**, 397–419 (1954).

[For part I see J. Research Nat. Bur. Standards **50**, 203–208 (1953); these Rev. **14**, 1099.] The random variables X_n, $n=0, 1, \cdots$, form a Markov chain with denumerable states and temporally homogeneous transition probabilities $P_{ij}^{(n)}=P(X_{m+n}=j \mid X_m=i)$; $i, j=0, 1, \cdots$. All states are in the same recurrent class. Let

$$F_{ij}^{(n)}=P(X_n=j, X_\nu\neq j, 1\leq \nu<n \mid X_0=i),$$

$$m_{ij}^{(p)}=\sum_{n=1}^{\infty} n^p F_{ij}^{(n)}, \quad p>0.$$

Theorem: $m_{ij}^{(p)}<\infty$ and $m_{ji}^{(p)}<\infty$ for some pair i, j implies $m_{kl}^{(p)}$, every k, l. A partial version for integral p, was proved by Hodges and Rosenblatt [Pacific J. Math. **3**, 127–136 (1953); these Rev. **14**, 886]. Let

$$E_{ij}=\lim_{n\to\infty} \sum_{\nu=0}^{n} P_{ij}^{(\nu)} \Big/ \sum_{\nu=0}^{n} P_{ii}^{(\nu)}.$$

For a real function f set $I(f)=\sum_j E_{ij}f(j)$. If $I(|f|)<\infty$, $I(|g|)<\infty$, $I(g)\neq 0$ (the choice of i here and later is immaterial), it is shown

$$P\left\{\lim_{n\to\infty} \sum_{\nu=0}^{n} f(X_\nu) \Big/ \sum_{\nu=0}^{n} g(X_\nu)=I(f)/I(g)\right\}=1.$$

[A proof using ergodic theory was given independently for certain general-state Markov processes by Harris and Robbins [Proc. Nat. Acad. Sci. U. S. A. **39**, 860–864 (1953); these Rev. **15**, 140]. Let $Y=f(X_{\nu_1})+\cdots+f(X_{\nu_2})$, where $X_{\nu_1}=i$ and ν_2 is the time of first return to i after ν_1. If $E(|Y|)<\infty$ and $m_{ii}^{(1)}<\infty$, the weak law of large numbers holds for $S_n=f(X_0)+\cdots+f(X_n)$. This is not sufficient for the strong law, as an example shows, but $m_{ii}^{(1)}<\infty$ and $\sum_{j=0}^{n}|f(j)|/m_{ij}^{(1)}<\infty$ together imply the strong law. Doeblin's central limit theorem is treated; if $m_{ii}^{(2)}<\infty$, then $E(Y^2)<\infty$ implies the central limit theorem, but an example shows $\sum_{j=0}^{n}(f(j))^2/m_{ij}^{(1)}<\infty$ does not. The law of the

iterated logarithm and the distribution of the maximum partial sum are treated. *T. E. Harris.*

Dobrušin, R. L. Conditions of regularity of Markov processes with a finite number of possible states. Mat. Sbornik N.S. 34(76), 541–556 (1954). (Russian)

Let $p_{ij}(s, t)$ be the transition probability of a Markov chain, the probability of a transition from state i at time s to state j at time t. There are finitely many states, and the parameter range is a compact interval. The transition probability function is called regular if, for every initial time and initial state, there is a corresponding Markov chain, with the given transition probabilities, whose sample functions are continuous except for jumps. The following theorems are proved. (1) If $\sum_j [1-p_{ij}(t_j, t_{j+1})] \leqq V_i < \infty$ (for all subdivisions of the parameter interval), then the transition probability function is regular, and for any corresponding chain, as described above, the expected number of jumps is at most $\sum_i V_i$. (2) Let $V_i(t)$ be defined like V_i except that the part of the parameter interval beyond the value t is deleted. Then the transition probability function is regular if and only if, whenever $V_j(t-0) = \infty$ [$V_j(t+0) = \infty$], for some values of j, t, it follows that $\lim_{h \downarrow 0} p_{kj}(s, t-h) = 0$ for $s < t$ [$\lim_{h \downarrow 0} p_{kj}(s, t+h) = 0$ for $s \leqq t$] and all k. Examples are constructed which exhibit the various possibilities.

J. L. Doob (Urbana, Ill.).

Ghurye, S. G. Random functions satisfying certain linear relations. Ann. Math. Statistics 25, 543–554 (1954).

Let $x(t)$ be a random process with values in p-dimensional Euclidean space, continuous in probability for $t \geqq t_0$. Let $A(h)$ be a $p \times p$ matrix function, continuous for $h \geqq 0$, nonsingular for some $h > 0$, and such that

$$(*) \quad x(t_0),\ x(t_0+mh) - A(h)x(t_0+(m-1)h) \quad (m = 1, \cdots, n)$$

are mutually independent for each $h > 0$ and $n \geqq 1$. It is supposed that $x(t_0)$ is not confined, with probability one, to any hyperplane of dimension $< p$. Then $A(h) = \exp(h\Lambda)$, Λ being a constant matrix, and there exists a process, $z(t)$, continuous in probability, with independent increments, such that

$$(**) \quad x(t) = \exp((t-t_0)\Lambda)x(t_0) + \int_{t_0}^{t} \exp((t-s)\Lambda)\,dz(s),$$

the integral being the limit in distribution of the Riemann-Stieltjes sums. Conversely, given such a $z(t)$ and a matrix, Λ, (**) satisfies (*). The logarithmic characteristic function of $x(t)$ is computed from that of $z(t)$. The process, $x_0(t) = \int_{t_0}^{t} \exp((t-s)\Lambda_0)\,dx(t)$, Λ_0 being a new constant matrix, is considered. *H. P. McKean Jr.*

Lévy, Paul. Rectification à un théorème sur le mouvement brownien à p paramètres. C. R. Acad. Sci. Paris 238, 2140–2141 (1954).

For each point A of p-dimensional Euclidean space, let $x(A)$ be a random variable. The family of differences $x(A) - x(B)$ is to be a Gaussian stochastic process with zero means and $E\{[x(A)-x(B)]^2\} = \bar{A}\bar{B}$. (Brownian motion with a p-dimensional parameter space.) The conditional expectation of $x(0)$ for $x(A)$ given on the surface S of the unit sphere with center 0 is the average z of the values of $x(A)$ on S. The author corrects his previous computation [Processus stochastiques et mouvement brownien, Gauthier-Villars, Paris, 1948; these Rev. 10, 551] of the variance of $x(0) - z$, finding that it approaches $1 - 2^{-1/2}$, rather than 0, when $p \to \infty$. This means, if A is now a point of Hilbert space, that the corresponding variance does not vanish.

J. L. Doob (Urbana, Ill.).

Feller, William. Diffusion processes in one dimension. Trans. Amer. Math. Soc. 77, 1–31 (1954).

The present paper has as its main purpose a probabilistic discussion of the general diffusion process in one dimension. Starting with the backward diffusion equation

$$(1) \quad u_t(t, x) = \Omega u(t, x), \quad \Omega = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx},$$

in a finite or infinite interval $r_1 < x < r_2$, a diffusion process obeying (1) is defined to be a stationary Markov process for which the Laplace transform of the transition probabilities at λ is a Green's function for the operator $\lambda I - \Omega$. In a previous paper [Ann. of Math. (2) 55, 468–519 (1952); these Rev. 13, 948] the author obtained the totality of all positive contraction semi-groups of operators on the Banach space of functions continuous in $[r_1, r_2]$ for the backward diffusion equation. This required a discussion of the most general kinds of lateral conditions compatible with the problem. These lateral conditions are now derived independently from probabilistic considerations.

A Markov process can be interpreted as a description of an ensemble of path functions $[X(t)]$. The author assumes throughout that almost all path functions in a diffusion process are continuous, except perhaps at the boundaries. As a consequence there exists a well-defined probability that a path starting at x will reach a point ρ_2 without first crossing ρ_1. These first passage probabilities are derived from the renewal principle and are the same for all processes obeying (1) for interior points. Thus (1) determines a class of processes which are not distinguishable as long as the boundary is not reached. A boundary point which can be reached in finite time from an interior point is called accessible; otherwise it is called inaccessible. A necessary and sufficient condition that there exist one and only one process satisfying (1) is that both boundary points be inaccessible. If at least one boundary is accessible then there are infinitely many processes connected with (1). The simplest of these is the "absorbing barrier process" defined by the rule that the process terminates at the moment the boundary is reached. More generally we have the "elementary return process" defined by the rule: If at time t the boundary r_j is reached, then $X(s) = r_j$ for $t \leqq s < t+T$, where T is a random variable independent of the past with $\Pr\{T > t\} = \exp(-t/\sigma_j)$; at time $t+T$ a jump occurs to a point x_j in the interior or to the boundary r_k with $\Pr\{X(t+T) = r_k\} = p_{jk}$, $\Pr\{r_1 < X(t+T) \leqq x\} = \tau_j p_j(x)$, after which the process starts over; here $p_{j1} + p_{j2} + \tau_j \leqq 1$, $p_{jk} \geqq 0$, $\tau_j \geqq 0$, $\sigma_j \geqq 0$, and $p_j(x)$ are monotonic functions with $p_j(r_1+) = 0$ and $p_j(r_2-) = 1$. Still another type of behavior at the boundary is that associated in classical diffusion theory with elastic or reflecting barriers. The author characterizes such processes by a passage to the limit with instantaneous ($\sigma_j = 0$) return processes. Finally the most general diffusion process associated with (1) is obtained by starting with the elastic barrier process and superimposing on it an elementary return process. By directly computing the Laplace transform of the transition probabilities for these processes, the author obtains an independent derivation of the resolvent operators for Ω in the case of all semi-groups obtained in the earlier paper. The corresponding

operators for the forward diffusion equation are obtained as before by determining the adjoint operator to Ω in the Banach-space sense. It is only in the simple absorbing and elastic barrier cases that the forward diffusion equation turns out to be the familiar formal adjoint operator. The general forward diffusion equation is no longer a differential equation. *R. S. Phillips* (Stanford, Calif.).

Kitagawa, Tosio. Linear stochastic translatable functional equations and stochastic Cauchy series. Jap. J. Math. 22 (1952), 1–18 (1953).

The author extends to regular random functions (r.r.f.) certain results obtained for deterministic functions in his previous papers [Jap. J. Math. 13, 233–332 (1937); 14, 125–168 (1938); Mem. Fac. Sci. Kyūsyū Imp. Univ. A. 1, 1–28 (1940); these Rev. 2, 104]. A complex-valued function $x=x(t,\omega)$ on $T\times\Omega$, where T is $(-\infty,\infty)$ and Ω a probability space, belongs to the class $\mathfrak{R}$ of r.r.f.'s when the expectation $E_\omega\{|x(t,\omega)|^2\}<\infty$ for almost all t. Let $\bar{x}(t)=E_\omega\{x(t,\omega)\}$ and $\{\sigma(t)\}^2=E_\omega\{|x(t,\omega)-\bar{x}(t)|^2\}$. For any bounded measurable subset M of T the author writes

$$\|x\|_M=\left(\int_M|\bar{x}(t)|^p\right)^{1/p}+\left(\int_M\{\sigma(t)\}^p\right)^{1/p}$$

and says that $x\in\mathfrak{L}^p\mathfrak{R}$, $1\leqq p<\infty$, when $\|x\|_M<\infty$ for every M, the integrals existing in Lebesgue's sense: similarly for $p=\infty$ with $\bar{x}(t)$ and $\sigma(t)$ continuous. He considers linear translatable operations Λ_t in $\mathfrak{L}^p\mathfrak{R}$, that is, (i) Λ_t transforms each element of $\mathfrak{L}^p\mathfrak{R}$ linearly into another such element, (ii) Λ_t is commutative with all translations of t and with E_ω, (iii) Λ_t is bounded, that is, $\|\Lambda_t x\|\leqq C\|x\|_{M'}$ where C and M' depend only on Λ_t and M. Then $\Lambda_t e^{\lambda t}=G_\Lambda(\lambda)e^{\lambda t}$, where $G_\Lambda(\lambda)$ is an integral function with zeros, say λ_n, of multiplicities p_n $(n=1,2,\cdots)$. The author calls a formal series

$$\sum_{n=1}^{\infty}\sum_{k=0}^{p_n-1}a_{nk}(\omega)t^k e^{\lambda_n t},$$

where the $a_{nk}(\omega)$'s are random variables with finite variances, a random Cauchy series. He studies the relations between such series and functions x in $\mathfrak{L}^p\mathfrak{R}$, especially when x is a solution of the equation $\Lambda_t x=0$. In this case he shows that suitable contour integrals $T_r(x)=T_r(t,x)$ $(r=1,2,\cdots)$, formed from x, reduce to partial sums of a random Cauchy series, and that (Theorem B) $\|x-T_r(x)\|_M\to 0$ $(r\to\infty)$ for every M. There are many misprints: in particular, in Lemma 2.2 delete (2°); in Lemma 5.3, (5.07), either read x for $\bar{x}$ throughout, or for $\bar{x}(u,\omega)$ and $\bar{x}(\eta,\omega)$ read $\bar{x}(u)$ and $\bar{x}(\eta)$. *H. P. Mulholland* (Birmingham).

Itô, Kiyosi. Complex multiple Wiener integral. Jap. J. Math. 22 (1952), 63–86 (1953).

A complex normal system is a family of complex-valued random variables such that every linear combination (complex coefficients) of the random variables has a normal distribution, with mean zero, and independent identically distributed real and imaginary parts. Equivalently, all joint distributions are Gaussian, with zero means, and each random variable is orthogonal to the conjugate of itself and of every other random variable of the family. [See Doob, Stochastic processes, Wiley, New York, 1953; these Rev. 15, 445, for a discussion of such families.] The covariance function of such a system determines the joint distributions, and orthogonality implies independence. To each subset of finite measure mA of a measure space, the author makes correspond a random variable $M(A)$ of a complex normal system, in such a way that $E\{M(A)\overline{M}(B)\}=m(A\cap B)$. Integrals of the form

$$\int\cdots\int f(t_1,\cdots,t_p,s_1,\cdots,s_q)\times dM(t_1)\cdots dM(t_p)d\overline{M}(s_1)\cdots d\overline{M}(s_q)$$

are then defined, and their properties derived. The results correspond to those in the real case, treated in a previous paper [J. Math. Soc. Japan 3, 157–169 (1951); these Rev. 13, 364]. As application, it is shown, among similar theorems, that the shift transformation group, acting on a stochastic process with stationary increments, which is a complex normal system, is ergodic if and only if the corresponding spectral distribution function is continuous. The corresponding theorem for stationary processes, due to Maruyama [Mem. Fac. Sci. Kyūsyū Univ. A. 4, 45–106 (1949); these Rev. 11, 257], is easily deduced from this theorem. *J. L. Doob* (Urbana, Ill.).

Kawata, Tatsuo. Remarks on prediction problem in the theory of stationary stochastic processes. Tôhoku Math. J. (2) 6, 13–20 (1954).

The author discusses the linear prediction problem of stationary (wide sense) stochastic processes, under the hypothesis that the process has no deterministic component and that the spectral distribution has a finite moment of order $2p$, where p is a positive integer. Then, under a supplementary hypothesis, he finds that the best linear prediction of $X(t+a)$, in terms of the past up to time t, has the form $\sum_0^{p-1}(a^j/j!)X^{(j)}(t)$ plus a generalized weighted average of $X(s)$, for the values of $s<t$. This result is, without rigorous details, contained in Wiener, "Extrapolation, interpolation, and smoothing of stationary time series" [Wiley, New York, 1949; these Rev. 11, 118]. *J. L. Doob* (Urbana, Ill.).

Takano, Kinsaku. Note on Wiener's prediction theory. Ann. Inst. Statist. Math. Tokyo 5, 67–72 (1954).

Let $\{X(t,\omega);-\infty<t<\infty,\omega\in\Omega\}$ be a measurable, strictly stationary, ergodic, stochastic process defined on the probability space Ω with $E\{|x(0)|^2\}<\infty$ and $x(t)$ continuous in mean. Then

$$R(u)=E\{x(t+u)\overline{x(t)}\}=\int_{-\infty}^{\infty}e^{iux}dF(x)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}x(t+u,\omega)\overline{x(t,\omega)}dt$$

for all u and almost all ω. If $f(t)$ is a sample function of a process for which the spectral distribution function $F(x)$ is known, Wiener defined the operator $\int_0^\infty f(t-\tau)dK(\tau)$ to be optimal for predicting $f(t+\alpha)$ if

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left|f(t+\alpha)-\int_0^\infty f(t-\tau)dK(\tau)\right|^2dt$$

is a minimum. The author remarks that it is not clear whether this last limit exists and whether it attains its minimum [$K(t)$ is of finite total variation] when it does exist. Set

$$\Delta(n,\tau_j,\alpha_j)=\left\{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left|f(t+\alpha)-\sum_{j=1}^{n}\alpha_j f(t-\tau_j)\right|^2dt\right\}^{1/2}$$

and $\delta=\inf_{n,\tau_j,\alpha_j}\Delta(n,\tau_j,\alpha_j)$, where n runs over all positive integers, the τ's over the non-negative real numbers, and the

α's over the complex numbers. He studies the problem of finding n, τ_j, α_j which satisfy $\Delta(n, \tau_j, \alpha_j) \leqq \delta + \epsilon$.

Early in the paper the author states that he does not assume $E\{x(t)\}$ independent of t, but this is an immediate consequence of strict stationarity. *R. A. Leibler.*

Brillouin, L. Information theory and uncertainty principle. J. Appl. Phys. **25**, 887–893 (1954).

The author continues his studies of the negentropy principle of information [same J. **24**, 1152–1163 (1953); **25**, 595–599 (1954)]. Let P_0 = initial number of equally probable possibilities before any information has been obtained, P_1 = final number of possibilities after information I has been obtained, k = Boltzmann's constant. Then I was given by $I = k \ln (P_0/P_1)$. It was shown that information corresponds to a negative term in the entropy system. The efficiency ϵ of the experimental method was defined by $\epsilon = \Delta I/\Delta S$ where ΔS is the increase in entropy. These concepts are applied to the problem of observations under a microscope to throw new light on the uncertainty relations $\Delta x \Delta p_x \sim h$. It is proved that under certain conditions $\Delta x \Delta p_x \sim Ah$, $A \gg 1$, actually obtains. The quantum uncertainties of Bohr-Rosenfeld are then examined by the same techniques and are found to be valid.

R. A. Leibler (Washington, D. C.).

Piesch, Johanna. Die Beanspruchung der Bündel im modernen Fernsprechverkehr. Arch. Elektr. Übertragung **8**, 324–328, 353–362, 411–419 (1954).

This is a summary report on telephone traffic problems associated with a simple trunk group of size n (the many-server problem of queue theory). Both "busy-signal" (no waiting lines) and waiting-line systems, the latter with and without defections in the lines, are considered, the work reported being mostly that of Conny Palm [chiefly, Special-nummer för Teletrafikteknik, Tekniska Meddelanden från Kungl. Telegrafstyrelsen, Stockholm, 1946, pp. 1–109]. The presentation is careful and thorough; no new results appear. The most interesting of Palm's material (loc. cit.) is that connected with the distribution of busy periods of waiting-line systems, that is, of times during which all trunks (servers) are occupied. For a single server with constant service time and a random (Poissonian) input, this distribution has been studied by Borel [C. R. Acad. Sci. Paris **214**, 452–456 (1942); these Rev. **4**, 248]; the generalization to an arbitrary service-time distribution is given, as an application of the theory of branching processes, by D. G. Kendall [J. Roy Statist. Soc. Ser. B. **13**, 151–185 (1951); these Rev. **13**, 957]. For exponential service time, the distribution is given as an integral involving the Bessel function I_1. Palm's result, which is for exponential service time only, is obtained by solution of the system of differential recurrence relations:

$$f_q'(t) = \gamma f_{q+1}(t) - (\gamma + n/s) f_q(t) + (n/s) f_{q-1}(t), \quad q = 0, 1, \cdots,$$

with $f_q(t) = 0$, $q < 0$, and $f_q(0) = 1$, all q. Here γ is the average incoming traffic density, n the number of servers, s the average service time and $f_q(t)$ the probability that a busy period which at a given epoch has a waiting line of length q continues as a busy period for at least t. A solution of these equations which agrees with Kendall for $f_0(t)$ and with the generalization of this, for $f_{q-1}(t)$, mentioned in discussion of Kendall's paper by I. J. Good has been obtained by S. O. Rice (unpublished). Finally it should be mentioned that the system of equations above is identical with that given by E. Vaulot [C. R. Acad. Sci. Paris **238**, 1188–1189 (1954); these Rev. **15**, 636] for "last come, first served" service delays, and that Vaulot's solution differs from both of those mentioned. *J. Riordan* (New York, N. Y.).

Ottaviani, Giuseppe. Sul problema della riassicurazione. Giorn. Ist. Ital. Attuari **15**, 65–84 (1952).

Consider a company i ($i = 1, 2, \cdots, n$) insuring n_i risks that are independent of one another and of the risks in each of the other companies. The mean and standard deviation of the jth risk are c_{ij} and σ_{ij} ($j = 1, 2, \cdots, n_i$), respectively, and a proportion α_{ij} ($0 < \alpha_{ij} \leqq 1$) of that risk is to be reinsured with company $i = n+1$. The author determines the "best" set of α_{ij} by minimizing $\sigma_0 + \sum_i \sigma_i$, where $\sigma_0^2 = \sum_{i,j} (1-\alpha_{ij})^2 \sigma_{ij}^2$ and $\sigma_i^2 = \sum_j \alpha_{ij}^2 \sigma_{ij}^2$, subject to an assigned overall quota of retention, viz. $\alpha_i \sum_j c_{ij}$, for company i. *H. L. Seal.*

Mathematical Statistics

Abdel-aty, S. H. Tables of generalized k-statistics. Biometrika **41**, 253–260 (1954).

Dans un article sur les k-statistiques généralisées $K_{r,\cdots}$, Wishart [Biometrika **39**, 1–13 (1952); ces Rev. **14**, 296] a montré que ces $K_{r,\cdots}$ s'expriment en fonction des "fonctions symétriques augmentées", et reciproquement, donnant les coefficients numériques de ces relations jusqu'aux fonctions de poids 6 inclus. L'auteur donne ces coefficients pour le poids 12; les coefficients pour les poids < 12 s'en déduisent.

R. Fortet (Paris).

Belz, Maurice H., and Hooke, Robert. Approximate distribution of the range in the neighborhood of low percentage points. J. Amer. Statist. Assoc. **49**, 620–636 (1954).

Let $Y_1, \cdots, Y_n$ be n independent and identically distributed chance variables, and let $X_1, \cdots, X_n$ denote the ordered values of $Y_1, \cdots, Y_n$, where $X_1 \leqq \cdots \leqq X_n$. Define $P_1(a)$ as $P(X_n - X_1 > a)$; define $P_2(a)$ as $P(W_n - W_1 > a)$, where W_1 and W_n are independently distributed, W_i having the same distribution as X_i ($i = 1, n$); define $P_3(a)$ as $P(Z_n - Z_1 > a)$, where Z_1 and Z_n are independently and normally distributed, Z_i having the same mean and variance as X_i ($i = 1, n$); and define $R(a)$ as $P_1(a)/P_2(a)$, and R as $\lim_{P_1(a) \to 0} R(a)$. The authors discuss the use of $P_2(a)$ and $P_3(a)$ as approximations to $P_1(a)$ for values of a such that $P_1(a)$ is small. They show that if the common distribution $Y_1, \cdots, Y_n$ has a finite range, $R = (n-1)/n$, so that the indicated approximation to $P_1(a)$ is $((n-1)/n)P_2(a)$. For distributions with just one infinite tail, R lies between $(n-1)/n$ and 1. The authors compute $P_1(a)$, $P_2(a)$, $P_3(a)$, $R(a)$ and R for various distributions and values of a. The possibility of using similar approximations for the distribution of other linear functions of $X_1, \cdots, X_n$ is discussed.

L. Weiss (Charlottesville, Va.).

Laha, R. G. On some properties of the Bessel function distributions. Bull. Calcutta Math. Soc. **46**, 59–72 (1954).

A random variable is said here to be a Bessel variate if it obeys a frequency function of the form

$$Cx^{(p-1)/2} e^{-\alpha x} I_{p-1}(\beta x^{1/2}), \quad 0 \leqq x < \infty,$$

in which $I_{p-1}(\beta x^{1/2})$ is the modified Bessel function of the first kind (Watson). By the use of characteristic functions, the author finds the frequency functions of a number of

functions of random variables. In Section I he finds the distribution of a sum of independent Bessel variates and from this he obtains the known distributions of a sample mean from a Bessel population, of a mean from a gamma population, of a non-central χ^2 for which he also derives its known reproductive property. In Section II he gets the distribution of the ratio of two independent Bessel variates and from this the distributions of the ratio of two non-central χ^2's and of a non-central χ^2 to a central χ^2. Again some of his results were known. In Section III he obtains the distribution of the difference of two independent Bessel variates which he applies to the product of two independent normal deviates one of whose means is zero, to the difference of two gamma variates, and to the difference of a Bessel variate and a gamma variate. The first two of these special cases have been previously given. In Section IV he finds the distribution of a linear form of independent Bessel variates which he successfully applies to a linear form in gamma variates and to a positive definite non-central quadratic form in normal variables. Finally, in Section V he derives the distribution of the ratio of two independent linear forms in Bessel variates and applies his result to the ratio of independent linear forms in gamma variates. *C. C. Craig.*

Tiago de Oliveira, J. Composite distributions and its application to some ecological problems. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) **3**, 171–175 (1954).

Consider two initial discrete distributions, whose chief parameters are the mean number of nodulus per ecological quadrat ν and the mean number of plants by nodulus λ respectively. It is desired to estimate $\mu=\nu\lambda$ (the mean number of individuals in a quadrat) or ν and λ separately. The author presents a simple method of doing this by observing the number of quadrats having 0, 1, or more than 1 individual in a quadrat. *L. A. Aroian.*

Masuyama, Motosaburo. On the unification of two approximations of a non-central F-distribution. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. **3**, 55–56 (1954).

Two different approximations for percentage points of the non-central F distribution, given previously by the author [same Rep. **1**, no. 4, 1–6 (1952); these Rev. **14**, 667], are related in terms of an approximation for the non-central chi-square distribution due to Patnaik [Biometrika **36**, 202–232 (1949); these Rev. **11**, 608]. *T. E. Harris.*

Ruben, H. On the moments of order statistics in samples from normal populations. Biometrika **41**, 200–227 (1954).

The sth moment of the rth order statistic in a sample of size n from a standard normal distribution is

$$(1)\qquad {}_s\mu'_{r|n}=n\binom{n-1}{r-1}\phi(s;1;r-1,n-r),$$

$\phi(s;\alpha;\beta,\gamma)$ being equal to $\int_{-\infty}^{\infty}x^{\alpha}z^{\beta}p^{\beta}q^{\gamma}dx$, where

$$z=[\exp(-\tfrac{1}{2}x^2)]/(2\pi)^{1/2}=\varphi(x),$$

$$p=\int_{-\infty}^{x}\varphi(\xi)d\xi,\quad q=1-p,\quad \alpha>0,$$

and s, β, γ are non-negative integers. Using the operators A, B, Γ which change α, β, γ into $\alpha+1$, $\beta-1$, $\gamma-1$, and the operators $P=\alpha^{-1}A$, $H=\beta B$, $K=\gamma\Gamma$, $L=H-K$, the author first proves

$$(2)\qquad \phi(2k+\delta;\alpha;\beta,\gamma)=\sum_{i=0}^{k}a_{2k+\delta,2i+\delta}(\alpha)(PL)^{2i+\delta}\phi(0;\alpha;\beta,\gamma);$$
$$k=0,1,2,\cdots;\ \delta=0,1.$$

The coefficients $a_{\beta,\gamma}(\alpha)$ are given for $\beta=0(1)10$ both for general α and for $\alpha=1$. Next the author proves

$$(3)\qquad \phi(0;\alpha;\beta,\gamma)=[V_{\beta,\beta+\gamma}(\cos^{-1}-(1+\alpha)^{-1})]/[(2\pi)^{\alpha-1}\alpha]^{1/2},$$

where $V_{\beta,\beta+\gamma}(\theta)$ denotes the ratio of the area of a (hyper-) spherical simplex in $\beta+\gamma$ dimensions to that of the corresponding unit sphere, the simplex being demarcated by two sets of (hyper-) planes, comprising β and γ planes respectively, and the angle between any two planes being θ, if they belong to the same set, and $\pi-\theta$ if they belong to opposite sets. By means of (2) and (3) formula (1) is simplified into

$$(1')\qquad {}_{2k+\delta}\mu'_{r|n}=n\binom{n-1}{r-1}\sum_{i=0}^{k}a_{2k+\delta,2i+\delta}(1)\times\frac{L^{2i+\delta}V_{r-1,n-1}(\cos^{-1}-(2i+\delta+2)^{-1})}{(2\pi)^{\frac{1}{2}(2i+\delta)}(2i+\delta+1)^{\frac{1}{2}}[(2i+\delta)!]}.$$

Using Schläfli's differential recurrence relation for the area of a general (hyper-) spherical simplex [L. Schläfli, Ges. math. Abh., Bd I, Birkhäuser, Basel, 1950; these Rev. **11**, 611] and the transformation $\theta=\cos^{-1}(-x^{-1})$ the author obtains the recurrence formula

$$(4)\qquad \bar{V}_{\beta,\beta+\gamma}(x)\equiv V_{\beta,\beta+\gamma}(\theta)=\frac{1}{2^{\beta+\gamma}}+\frac{1}{4\pi}L^{\beta}\int_{x}^{\infty}\bar{V}_{\beta,\beta+\gamma}(x'+2)\frac{dx'}{x'(x'^2-1)^{1/2}}.$$

The author gives tables of $\bar{V}_{\beta,\beta}(x)$ for $x=2(1)12$ and $\beta=0(1)51-x$. For $r=n$ very accurate tables of the first 10 moments, the moments about the mean of orders 2–4, the standard deviation, skewness and kurtosis are given for $n=1(1)50$. *D. M. Sandelius* (Göteborg).

Cadwell, J. H. The statistical treatment of mean deviation. Biometrika **41**, 12–18 (1954).

Let $\bar{m}(k,n)$ be the average of k mean deviations, each for a random sample of size n from a normal population with standard deviation σ. Then the quantity $c\{\bar{m}(k,n)/\sigma\}^{1.5}$ is found to be approximately distributed as χ^2 with ν degrees of freedom, where c and ν depend on k and n. Tables of c and ν are given for $k=1(1)10$, $n=4(1)10$ and for $k=1(1)5$, $n=15(5)50$; these give also the expected value and variance of $\bar{m}(1,n)/\sigma$ for $n=1(1)10(5)50$. There is also a table of lower and upper 2.5% and 5% points of $\bar{m}(k,n)/\sigma$ for $k=1(1)10$, $n=4(1)10$. *D. M. Sandelius* (Göteborg).

Huybrechts, M. La borne inférieure de la variance d'une estimation. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **40**, 791–797 (1954).

Let $x_1,\cdots,x_n$ be random variables with a joint probability density F depending on a parameter θ. Let t be a function of $x_1,\cdots,x_n$. Under certain regularity conditions it is known that

$$E(t-\theta)^2\geqq(d\psi/d\theta)^2\Big/E\left[\left(\frac{\partial\log F}{\partial\theta}\right)^2\right],$$

where $E(t)=\psi(\theta)$. The author shows that the inequality is still true if the left side is replaced by the variance of t.

T. E. Harris (Santa Monica, Calif.).

Sugiyama, Hiroshi. Some theory of control charts. I. Math. Japonicae **3**, 13–29 (1953).

The author's main idea is that the population mean, m, in quality control is not a constant but a random variable with standard deviation σ^*, while the standard deviation within lots is σ. Symbols are used without proper definition and many misprints occur. With this particular model the

author investigates the risks inherent in quality control charts and suggests various choices of limits. The results are extended to attributes. The reviewer feels that nothing is gained by such a model but much is lost and the customary one where m is constant is the proper model, as explained in the paper of the reviewer and Levene [J. Amer. Statist. Assoc. 45, 520–529 (1950); erratum 47, 685 (1952); these Rev. 15, 142]. *L. A. Aroian* (Culver City, Calif.).

Sugiyama, Hiroshi. Some theory of control charts. II. (Summarized preliminary report). Math. Japonicae 3, 30–31 (1953).

Summary of further results along the lines indicated in the paper reviewed above. *L. A. Aroian.*

Sugiyama, Hiroshi. Some theory of control charts. III. (Summarized preliminary report). Math. Japonicae 3, 33–35 (1953).

Application of the author's theory to control of a variable by a correlated variable. [See the two preceding reviews.] *L. A. Aroian* (Culver City, Calif.).

Masuyama, Motosaburo. Graphical method of statistical inference. IV. Mixing ratio of particles of two kinds. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. 3, 1–2 (1953).

[For part III see same Rep. 1, no. 4, 1–6 (1952); these Rev. 14, 667.] A method is described, using the author's improved binomial probability paper [ibid. 1, no. 2, 15–22 (1951); these Rev. 13, 961] for estimating the relative proportions of two kinds of objects in a mixture, based on a sample. A bacteriological application is indicated. *T. E. Harris* (Santa Monica, Calif.).

Masuyama, Motosaburo. Graphical method of statistical inference. V. On a Gamma distribution and related problems. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. 3, 3–5 (1953).

The author gives additional uses for binomial probability paper [same Rep. 1, no. 2, 15–22 (1951); these Rev. 13, 961]: 1) an approximation for percentage points of gamma distributions; 2) confidence intervals for the variance in a normal population; 3) tests of hypotheses concerning the unknown variance of a normal distribution. *T. E. Harris* (Santa Monica, Calif.).

Mood, A. M. On the asymptotic efficiency of certain nonparametric two-sample tests. Ann. Math. Statistics 25, 514–522 (1954).

The asymptotic relative efficiency of several nonparametric two-sample tests with respect to the corresponding standard parametric tests in case of normality is given. In particular, it is shown that the asymptotic relative efficiency of the rank test of dispersion based on the statistic $W=\sum_i[r_i-\frac{1}{2}(m+n+1)]^2$, r_i being the ranks of the observations in one of the two samples in the overall ranking, is $15/2\pi^2$. The author does not give a clear definition of what he understands by the asymptotic efficiency of a test. This omission might easily lead to misunderstandings in connection with the author's discussion of one- and two-sided tests. *G. E. Noether* (Boston, Mass.).

Basu, D. On the optimum character of some estimators used in multistage sampling problems. Sankhyā 13, 363–368 (1954).

Let $u_1, \cdots, u_M$ be primary sampling units; θ_j is a number representing a characteristic of u_j; it is required to estimate $\theta_1+\cdots+\theta_M$. If u_j is used, the estimator t_j of θ_j is predetermined. Optimum selection of a predetermined fixed number m of the u_j and estimation of the variance of the minimum-variance estimator is determined for the following set-ups. (A) Selection is with replacement with (possibly) different probabilities of selecting the different u_j; if a particular unit appears r times, r independent estimates for that unit are used. (B) Selection is without replacement, with equal probabilities for each primary unit. (C) As (A), except that if a particular unit appears r times, the same estimate is used r times. *T. E. Harris.*

Bahadur, R. R. Sufficiency and statistical decision functions. Ann. Math. Statistics 25, 423–462 (1954).

This paper is both a contribution to and a self-contained critical review of the abstract study of sufficient statistics initiated by P. R. Halmos and the reviewer [same Ann. 20, 225–241 (1949); these Rev. 11, 42], and carried forward by several other authors.

The author sets himself the two main tasks of showing "that the justification for the use of sufficient statistics in statistical methodology which is sketched in the final section of [the paper cited above] . . . is valid under quite general conditions, and to extend this justification to the case of sequential experiments." For the first of these justifications, it proves to be enough that the decision space (not the sample space) be σ-isomorphic to the Borel line, which is almost no restriction at all in practice. For the second, where a decision to continue observation is among the available decisions, if a sufficient statistic is to do all that is desired of it, it must be sufficient not only for the unknown parameter but also for future observations, so to speak. The author fully formulates this subsidiary condition, and explores and simplifies it. He shows that the subsidiary condition is automatically fulfilled in the case of sequential observation of independent observations, the common case in practice. New light is cast on minimal sufficient statistics, on the relative roles of subfield and statistic, and on pathology and how to avoid it. *L. J. Savage* (Chicago, Ill.).

Mann, H. B. A theory of estimation for the fundamental random process and the Ornstein Uhlenbeck process. Sankhyā 13, 325–350 (1954).

Mann, Henry B., and Moranda, Paul B. On the efficiency of the least square estimates of parameters in the Ornstein Uhlenbeck process. Sankhyā 13, 351–358 (1954).

Soient, sur l'intervalle $(0\leqq t<+\infty)$, $X(t)$ un processus aléatoire, et $\phi_1(t)$, $\phi_2(t)$, $\cdots$, $\phi_s(t)$ s fonctions certaines indépendantes; soit $f(t)$ la fonction: $f(t)=\sum K_j\phi_j(t)$, où les K_j sont inconnus; $X(t)$ est d'un type déterminé, mais un ou plusieurs paramètres de sa loi de probabilité peuvent être inconnus. On observe (complètement) sur $(0, T)$ le processus $Y(t)=X(t)+f(t)$. Le problème est d'en déduire une estimation des paramètres inconnus, ou des K_j, ou des deux à la fois. Les auteurs examinent le cas où $X(t)$ est un processus de Wiener-Levy, et celui où $X(t)$ est un processus de Ornstein-Uhlenbeck. Il serait difficile de résumer ici leurs résultats; ils envisagent l'application de la méthode du maximum de vraisemblance, et de celle des moindres carrés, étudient en particulier si, lorsque $T\to+\infty$, les estimations obtenues tendent en probabilité vers les valeurs vraies des quantités à estimer. En prenant comme $\phi_j(t)$ les fonctions $\cos 2\pi jt$, $\sin 2\pi jt$, et en faisant $S\to+\infty$, on peut estimer $f(t)$ dans le cas où cette fonction est une fonction periodique continue à variation bornée quelconque. *R. Fortet.*

Hoel, Paul G. Confidence bands for polynomial curves. Ann. Math. Statistics **25**, 534–542 (1954).

The author considers the construction of confidence bands for polynomial growth-type curves. If the values of several sample curves are known at a number of points then a T statistic (distribution supposed known) may be calculated from this multivariate sample, and a value T_0 determined such that $T>T_0$ with probability C_0. The author now gives a procedure for calculating the upper and lower limits of the expected curve at a given time point which would be consistent with the facts that $T>T_0$ and that the expected curve is a polynomial of given order. Amalgamating these limits for all time values, one obtains a confidence band whose content is evidently $\geqq C_0$. *P. Whittle.*

Del Chiaro, Adolfo. Sulla determinazione delle probabilità di eliminazione. Giorn. Ist. Ital. Attuari **15** (1952), 235–249 (1953).

This article purports to be a critical review of some of the estimators proposed for $q_\alpha(x, x+t)$ the probability that an individual aged x is eliminated from a community before age $x+t$ on account of cause α, one of several present. However, the author writes as if observational data can always be substituted for their expectations with the result that some of his preferred estimates (e.g. formula [15]) are not even unbiased in the limit. *H. L. Seal.*

Mathematical Economics

Beale, E. M. L. An alternative method for linear programming. Proc. Cambridge Philos. Soc. **50**, 512–523 (1954).

This paper presents a new computational procedure, the method of leading variables, to solve the linear programming problem. This method is compared with the simplex method of Dantzig [Activity analysis of production and allocation, Wiley, New York, 1951, pp. 339–347; these Rev. 15, 47] and the dual simplex method of Lemke [Carnegie Inst. of Tech., Dept. of Math., Tech. Rep. No. 29 (1953)]. The problem is to minimize a linear form subject to constraints in the form of simultaneous linear equations. The simplex method starts with a set of non-negative values satisfying the constraints, a feasible solution, and then by an iterative procedure finds a feasible solution which minimizes the linear form. The method of leading variables starts with a set of non-negative values which would minimize the linear form if it satisfied all the constraints. The constraints are then introduced in an iterative procedure until a solution is obtained. Many of the steps in the two methods are identical in form. The advantage in the method of leading variables lies in the simplification of the initial step.

E. D. Nering (Minneapolis, Minn.).

Bellman, Richard. On some applications of the theory of dynamic programming to logistics. Naval Res. Logist. Quart. **1**, 141–153 (1954).

This article considers multi-stage processes composed of a sequence of operations in which the outcome of the preceding operations may be used to guide the course of future operations. Examples of such processes can be found in many fields including industrial production, military planning, inter-industry analyses, and learning processes. A mathematical theory of dynamic programming is developed to treat such multi-stage processes arising in the field of logistics with examples of problems amenable to the techniques developed. (Author's summary.) *J. Wolfowitz.*

***Jensen, Arne. A distribution model applicable to economics.** Munksgaard, Copenhagen, 1954. 99 pp.

This book contains a treatment of some of the theoretical distributions encountered in combinatorial probabilities and their continuous analogs, and also brief accounts of some distributions used for empirical fitting. It also contains an elementary treatment of discrete and continuous queuing processes and applies this treatment to the formulation of models for certain economic problems. *H. Rubin.*

Nataf, André. Sur des questions d'agrégation en économétrie. Publ. Inst. Statist. Univ. Paris **2** (1953), no. 4, 5–61 (1954).

This monograph is a careful study of the aggregation problem, i.e., the problem of deducing from a model of economic relations involving individual economic agents and individual commodities, a model of relations among aggregates each of which is a function of several individuals' actions concerning several commodities. In the first part, he investigates the possibility of aggregation in the fields of production and consumption, separately, without further assumptions about the rest of the economy. In the field of production, the results maybe stated as follows: Let there be A firms, m types of output, r types of labor, and s types of capital. For each firm α, there is a transformation relation

$$F_\alpha(x_{1\alpha}, \cdots, x_{m\alpha}; n_{1\alpha}, \cdots, n_{r\alpha}; z_{1\alpha}, \cdots, z_{s\alpha})=0,$$

where $x_{i\alpha}$ is the output of commodity i by firm α, $n_{j\alpha}$ the use of type of labor j by firm α, and $z_{k\alpha}$ the use of type of capital k by firm α. The aggregation problem is then to define aggregates X, N, Z depending only on the $x_{i\alpha}$'s, $n_{j\alpha}$'s, and $z_{k\alpha}$'s, respectively, satisfying a condition $F(X, N, Z)=0$ if each of the transformation relations of the individual firms is satisfied. The problem is, of course, solvable only under special assumptions as to the form of the transformation relations $F_\alpha=0$; it is necessary and sufficient that they can be written in the form $G_\alpha+H_\alpha+I_\alpha=0$, where G_α depends only on $x_{1\alpha}, \cdots, x_{m\alpha}$, H on $n_{1\alpha}, \cdots, n_{r\alpha}$, and I_α on $z_{1\alpha}, \cdots, z_{s\alpha}$; in that case, $X=\sum G_\alpha$, $N=\sum H_\alpha$, and $Z=\sum I_\alpha$.

In the field of consumption, the results are of a somewhat different order; the aim is to find conditions under which the aggregated demand for each commodity is a function of prices and total revenue. The condition is that, for any fixed set of prices, the Engel curves of all individuals are straight lines with the same direction. Some geometric conditions on the indifference surfaces of the different individuals to insure this are stated.

In the second half an attempt is made to pose the problem of aggregation when some additional conditions are imposed on the variables to be aggregated. There are some results on the case of consumption of a group of individuals with equal incomes; a number of conditions are derived which do not lend themselves to easy summary. The remarks on aggregation of production in this section are more methodological than conclusive. *K. J. Arrow* (Stanford, Calif.).

Frisch, Ragnar. Linear expenditure functions. An expository article. Econometrica **22**, 505–510 (1954).

TOPOLOGY

***Kuratowski, Kazimierz. Elementy topologii. [The elements of topology.]** Państwowe Wydawnictwo Naukowe, Warsaw, 1953. 184 pp. zł. 14.00.

An introductory course in topology on a very popular level. A list of paragraph headings gives the best idea of the scope of this book. 1. Auxiliary notions from set theory. 2. Metric spaces. 3. Limits of sequences. 4. Closure. 5. Closed and open sets; boundary and interior. 6. Derivative of a set; dense sets. 7. Continuity. 8. Separability, bases. 9. Completeness. 10. Compactness. 11. Connectedness. 12. Continua. 13. Local connectedness. 14. Dimension. 15. The simplex. 16. Complexes, chains, homology. 17. Decomposition of the plane; Jordan's theorem. *S. Eilenberg.*

Zarankiewicz, K. On a problem of P. Turan concerning graphs. Fund. Math. **41**, 137–145 (1954).

A is a set of p points and B is a set of q points in the Euclidean plane. The two sets have no point in common. Every point of A is joined to every point of B by a simple arc lying in the plane and having the point of A and the point of B concerned as its two end points. No three arcs have a point that is not an end point of all of them in common. It is proved that there are at least $K(p, q)$ intersection points between the arcs, and the arcs divide the plane into at least $L(p, q)$ regions, where

$$K(2k, 2n) = (k^2-k)(n^2-n),$$
$$K(2k, 2n+1) = (k^2-k)n^2,$$
$$K(2k+1, 2n+1) = k^2n^2,$$
$$L(2k, 2n) = (k^2-k)(n^2-n)+4nk-2(n+k)+2,$$
$$L(2k, 2n+1) = (k^2-k)n^2+4nk-2n+1,$$
$$L(2k+1, 2n+1) = n^2k^2+4nk+1.$$

A graph is constructed for each pair of natural numbers p and q in which the number of intersection points of all the arcs is expressed exactly by the above formulae.

G. A. Dirac (Vienna).

Iséki, Kiyoshi. On the hannerization of completely normal spaces. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) **3**, 143–146 (1954).

Iséki, Kiyoshi. On Hannerisation of two countably paracompact normal spaces. Proc. Japan Acad. **30**, 443–444 (1954).

The following result, which was proved by O. Hanner for the case where all spaces involved are, respectively, normal, collectionwise normal, perfectly normal, or paracompact, is here established for the completely normal and normal-and-countably-paracompact case respectively: A space Y is an AR (resp. ANR) if and only if Y is an ES (resp. ENS). (The notation is that of Hanner.) [The term "Hannerization" may be misleading since, according to Hanner himself, the construction referred to is due to J. H. C. Whitehead.]

E. Michael (Seattle, Wash.).

Iséki, Kiyoshi. A note on retraction in completely normal spaces. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) **3**, 176–180 (1954).

The following theorem which was proved by N. Aronszajn and K. Borsuk for the cases where all spaces involved are separable metric or compact metric [Fund. Math. **18**, 193–197 (1932)], is here proved for the cases where all spaces are completely normal: If $X = X_1 \cup X_2$, X_1 and X_2 are closed in X, and $X_1 \cap X_2$ is an AR (resp. ANR), then X is an AR (resp. ANR) if and only if X_1 and X_2 both are AR (resp. ANR). *E. Michael* (Seattle, Wash.).

Nakano, Hidegorô. A generalization of Ascoli's theorem. Proc. Japan Acad. **30**, 282–284 (1954).

Soient R un ensemble, (E_λ), $\lambda \in L$, une famille d'espaces uniformes, et pour chaque $\lambda \in L$, soit $(f_{\alpha\lambda})$, $\alpha \in A_\lambda$, une famille d'applications de R dans E_λ. L'auteur considère sur R la structure uniforme la moins fine pour laquelle chacune des familles $(f_{\alpha\lambda})$, $\alpha \in A_\lambda$, soit uniformément équicontinue, et donne une condition moyennant laquelle R est précompact pour cette structure uniforme. *J. Dieudonné.*

Gillman, Leonard, and Henriksen, Melvin. Concerning rings of continuous functions. Trans. Amer. Math. Soc. **77**, 340–362 (1954).

The first part of this paper studies prime ideals in the ring $C(X, R)$ of all continuous, real-valued functions on a completely regular space X. It is shown, for instance, that every prime ideal in $C(X, R)$ is contained in a unique maximal ideal. Special attention is focused on spaces X such that every prime ideal in $C(X, R)$ is maximal. Such spaces are called P-spaces, and this property is shown to be equivalent to many others, such as: (a) every zero-set of an $f \in C(X, R)$ is open; (b) every G_δ in X is open; and (c) $C(X, R)$ is a regular ring.

The second part of the paper consists of a careful study of linearly ordered topological spaces (which are known to be normal). It is shown, among other things, that every linearly ordered space is countably paracompact [independently obtained by B. J. Ball, Proc. Amer. Math. Soc. **5**, 190–192 (1954); these Rev. **15**, 976], and that every linearly ordered Q-space [in the sense of E. Hewitt Trans. Amer. Math. Soc. **64**, 45–99 (1948); these Rev. **10**, 126] is paracompact. *E. Michael* (Seattle, Wash.).

Nachbin, Leopoldo. Topological vector spaces of continuous functions. Proc. Nat. Acad. Sci. U. S. A. **40**, 471–474 (1954).

Soit E un espace complètement régulier. L'auteur considère l'espace $\mathcal{C}(E)$ des fonctions continus numériques définies dans E, qui est un espace vectoriel localement convexe lorsqu'on le munit de la topologie de la convergence compacte; il démontre les deux théorèmes suivants: I. Pour que $\mathcal{C}(E)$ soit tonnelé, il faut et il suffit que pour toute partie fermée et non compacte X de E, il existe une fonction $f \in \mathcal{C}(E)$ non bornée dans X. II. Pour que $\mathcal{C}(E)$ soit bornologique, il faut et il suffit que E soit un Q-espace au sens de Hewitt, c'est-à-dire qu'il soit complet pour la structure uniforme la moins fine rendant uniformément continues les fonctions de $\mathcal{C}(E)$. Les espaces paracompacts satisfont à l'hypothèse de I, et l'hypothèse de II est vérifiée si tout recouvrement ouvert de E contient un recouvrement dénombrable de E. Par contre, l'auteur remarque qu'un espace donné en exemple par L. Gillman et M. Henriksen [voir l'analyse ci-dessus] vérifie l'hypothèse de I mais non celle de II, et il en conclut qu'il y a des espaces tonnelés non bornologiques. *J. Dieudonné* (Evanston, Ill.).

Kandô, Tetsuo. Characterization of topological spaces by some continuous functions. J. Math. Soc. Japan **6**, 45–54 (1954).

In Bull. Amer. Math. Soc. **59**, 180 (1953), the reviewer announced two results which characterize (1) paracompact

and (2) normal spaces by means of "selections" for lower semi-continuous set-valued functions from these spaces to families of closed, convex subsets of Banach spaces. In this paper, the author analogously characterizes spaces which are (3) normal and countably paracompact and (4) normal, and every point-finite covering has a locally finite refinement. [Reviewer's note: The results in the corollaries on page 52 can be strengthened by omitting "countably paracompact" in Corollary 1, "point-finitely paracompact" in Corollary 2, and "separable" in Corollary 3.]

E. Michael (Seattle, Wash.).

Novák, J. On a problem concerning completely regular sets. Fund. Math. **41**, 103–104 (1954).

A noncompact completely regular space is constructed on which all continuous real-valued functions are bounded, thus answering a question attributed to Słowikowski and Zawadowski. However, completely regular spaces on which all continuous real-valued functions are bounded have already been characterized completely by the reviewer [Trans. Amer. Math. Soc. **64**, 45–99 (1948), Theorem 28; these Rev. **10**, 126]. *E. Hewitt* (Seattle, Wash.).

Mrówka, S. On completely regular spaces. Fund. Math. **41**, 105–106 (1954).

A simplified form of Novák's construction [see the preceding review]. *E. Hewitt* (Seattle, Wash.).

Sikorski, R. Closure homomorphisms and interior mappings. Fund. Math. **41**, 12–20 (1954).

A closure algebra A [Sikorski, Fund. Math. **36**, 165–206 (1949); these Rev. **12**, 85] has three basic operations: enumerable addition, complementation, closure. A closure homomorphism of A into B is a function preserving these operations; also there is the concept of closure homeomorphism. If X is a topological space, $C(X)$ denotes the closure algebra of all subsets of X. If $f: X \to Y$ is continuous, then f is interior if and only if $f^{-1}: C(Y) \to C(X)$ is a closure homomorphism. If the cardinality of Y is less than the first aleph inaccessible in the strict sense, then Y is an interior image of X if and only if $C(Y)$ is homeomorphic with a closure subalgebra of $C(X)$. The aim of the paper is to study representations of closure algebras A with enumerable basis. It is proved that such an algebra A is weakly homeomorphic with a closure quotient algebra $C(X)/I$ where X is a T_0-space with an enumerable basis and I is a σ-ideal. It is conjectured that such a $C(X)$ is homeomorphic to a complete closure subalgebra of $C(Y)$ for some separable metric space Y. That is, it is conjectured that every T_0-space with an enumerable basis is an interior image of a separable metric space. *E. E. Floyd* (Charlottesville, Va.).

Ryll-Nardzewski, C. A remark on the Cartesian product of two compact spaces. Bull. Acad. Polon. Sci. Cl. III. **2**, 265–266 (1954).

Let $\mathfrak{m}$ be an infinite cardinal number. Let X be a countably compact topological space such that every point of X admits a complete family of open neighborhoods having cardinal number $\leq \mathfrak{m}$. Let Y be a topological space such that every open covering of Y of cardinal number $\mathfrak{m}$ admits a finite subcovering. Then $X \times Y$ is countably compact.

E. Hewitt (Seattle, Wash.).

Padmavally, K. A generalization of Cantor Bendixon theorem. Proc. Nat. Inst. Sci. India **20**, 305–306 (1954).

Let R be a locally compact Hausdorff space. The intersection character $\iota(p)$ at $p \in R$ is the least cardinal number m for which there exist m open subsets of R whose intersection is p. Let $\aleph_\alpha$ be a regular cardinal $> \sup_{p \in R} \iota(p)$, and let ω_α be the initial ordinal with corresponding cardinal $\aleph_\alpha$. For all $X \subset R$, it is proved that the ω_αth derived set of X is equal to the $(\omega_\alpha+1)$th derived set of X. This partially generalizes a classical theorem of Cantor and Bendixson [Cantor, Math. Ann. **23**, 453–488 (1884), Theorem F]. An example of a locally compact Hausdorff space R with $\iota(p) = \aleph_0$ for all $p \in R$ and containing a subset X such that the derived sets $X^{(\alpha)}$ of X form a strictly decreasing family for all $\alpha < \omega_1$ shows that the result cannot be materially strengthened. *E. Hewitt* (Seattle, Wash.).

Dowker, C. H. Inductive dimension of completely normal spaces. Quart. J. Math., Oxford Ser. (2) **4**, 267–281 (1953).

Let Ind denote that "large" inductive dimension. It is shown that, for a completely normal space X, $\bigcup_{i=1}^{n} A_i = X$, $\operatorname{Ind} A_i \leq n$ implies $\operatorname{Ind} X \leq n$ provided A_i are disjoint and $\bigcup_{i=1}^{n} A_i$ are closed. Consider the following properties of a topological space X: (α) if $B \subset A \subset X$, then $\operatorname{Ind} B \leq \operatorname{Ind} A$; ($\beta$) if $G \subset A \subset X$ and G is open in A, then $\operatorname{Ind} G \leq \operatorname{Ind} A$; ($\gamma$) if $B \cup C = A \subset X$ and B is closed, then $\operatorname{Ind} A \leq \max(\operatorname{Ind} B, \operatorname{Ind} C)$; ($\delta$) if $\bigcup_{i=1}^{n} A_i = A \subset X$ and A_i are closed in A, then $\operatorname{Ind} A \leq \sup \operatorname{Ind} A_i$. It is proved that, for a completely normal X, β implies α, γ, δ.

The author calls a normal space X totally normal (t.n.) if every open $G \subset X$ is the union of a locally finite (in G) collection of open F_σ sets of X. The t.n. spaces include the hereditarily normal Hausdorff spaces as well as the perfectly normal spaces; every t.n. space is completely normal; total normality is hereditary. It is proved that t.n. spaces possess properties α, β, γ, δ; this theorem extends older results of E. Čech [Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **33**, 38–55 (1932)]. The author announces an example of a normal Hausdorff space with $\operatorname{Ind} X = 0$ and $\operatorname{Ind} A = 1$ for some open $A \subset X$. *M. Katětov* (Prague).

Denjoy, Arnaud. Les couples de continus joints dans le plan. I, II, III. C. R. Acad. Sci. Paris **239**, 561–564, 654–657, 685–687 (1954).

Given two continua C and C' in a plane Π intersecting in a set Γ having p components, $p \geq 1$, and supposing that $C - \Gamma$ lies in a component R' of $\Pi - C'$ and $C' - \Gamma$ lies in a component R of $\Pi - C$, the author shows by elementary set-theoretic methods in these three notes that the intersection $R \cdot R'$ will consist of exactly p component regions. [For $p = 1$ the additional assumption that $\Pi - C$ and $\Pi - C'$ lie together in the same component of $\Pi - \Gamma$ is made.] The case $p = 1$ is treated in the first note, $p = 2$ in the second, and $p \geq 3$ in the third. Some indications of possible extension of this type of result to higher-dimensional euclidean spaces are given. *G. T. Whyburn* (Charlottesville, Va.).

Stein, S. Families of curves. Proc. Amer. Math. Soc. **5**, 745–747 (1954).

Let S_n be an n-sphere, E_{n+1} its interior, ϕ a topological transformation $S_n \to S_n$ of period $p \neq 1$. It is shown that every mapping $F: S_n \times [0, 1] \to E_{n+1}$ such that $F(P, 0) = P$, $F(P, 1) = F(\phi(P), 1)$ is a mapping onto E_{n+1}. Among the applications is a generalization of a theorem of Forrester [same Proc. **3**, 333–334 (1952); these Rev. **14**, 72] and a necessary and sufficient condition that a compact subset of R_n be a convex set having 1-point contact with each of its support planes. *P. A. Smith* (New York, N. Y.).

Weier, Josef. Sur la somme d'indices des coïncidences de deux représentations. C. R. Acad. Sci. Paris **239**, 609–610 (1954).

Let g, g' be mappings $Q \to R$ where Q, R are compact n-dimensional locally euclidean spaces. The author sketches an elementary proof of the theorem (implicit in the work of Lefschetz) that the sum of the indices of the coincidences of (g, g'), assumed finite in number, is a homotopy invariant. The author shows that it is sufficient to consider small homotopies. *P. A. Smith* (New York, N. Y.).

Tatarkiewicz, K. Les transformations unifoliées. Fund. Math. **41**, 122–136 (1954).

Let X and Y be euclidean spaces, let $M \subset X \times Y$, let φ: $M \to X \times Y$, and let π: $M \to X$ be the projection of M into X. The map φ is called one-leaved (unifolié) in case φ is one-to-one and the restriction to $\varphi(A)$ of π is one-to-one for every subset A of M such that the restriction to A of π is one-to-one. The author observes that in order that φ be one-leaved it is sufficient for the first coordinate of $\varphi(x, y)$ to be one-to-one in x and constant in y. The author proves that this condition is also necessary under certain additional assumptions. Such an assumption, for instance, is that M is open, the intersection of M and every hyperplane x=constant is connected, and the first coordinate of φ is a continuous function. Various examples are given which show that parts of the additional assumptions cannot be omitted. *W. H. Gottschalk* (Philadelphia, Pa.).

Hocking, John G. Approximations to monotone mappings on non-compact two dimensional manifolds. Duke Math. J. **21**, 639–651 (1954).

Let M be a 2-manifold contained in Euclidean 5-space and let f: $M \to N$ be a continuous function. (1) Let f be uniformly continuous, let $M = N = f(M)$, let M be ulc[1] (with or without ulc boundary). Then f may be uniformly approximated by homeomorphisms if and only if (A) the inverse of each compact set is compact and (B) $f^{-1}(y)$ is 0- and 1-acyclic for each $y \in N$. (2) Let M have boundary B, let $f|B$ be monotone and let (A) and (B) hold. Then $f(M)$ is homeomorphic with M. (3) Let M be compact with boundary B, let $f|B$ be monotone, let f be quasi-interior and let $f^{-1}(y)$ be 1-acyclic for each $y \in N$. Then $f(M)$ is a 2-manifold and f can be approximated by light interior maps. These results extend theorems of Youngs [same J. **15**, 87–94 (1948); these Rev. **9**, 524], Roberts and Steenrod [Ann. of Math. (2) **39**, 851–862 (1938)], and Martin [Duke Math. J. **8**, 136–153 (1941); these Rev. **3**, 141]. In addition to the principal theorems stated above there are numerous intermediary results and two sections of preliminary results involving inter alia some lemmas on regular convergence of sets. *A. D. Wallace* (New Orleans, La.).

Terasaka, Hidetaka. On quasi-translations in E^n. Proc. Japan Acad. **30**, 80–84 (1954).

A quasi-translation is a sense-preserving homeomorphism f of a Euclidean space E^n onto itself such that for each bounded subset M of E^n, $\limsup_{i \to \pm\infty} f^i(M) = 0$. According to a well known theorem [B. v. Kerékjártó, Ann. of Math. (2) **27**, 105–117 (1925); E. Sperner, Abh. Math. Sem. Hamburg. Univ. **10**, 1–48 (1934)] a quasi-translation is topologically equivalent to a translation in the ordinary sense when $n = 2$. Whether this is the case for $n > 2$ is an unsolved problem.

Let f be a homeomorphism of a sphere S^n onto itself with exactly one fixed point ξ and such that whenever M is a set with ξ non-ε $\bar{M}$, then $\lim_{i \to \pm\infty} f^i(M) = \xi$. The author shows that there exists an open polyhedron π with sole boundary at ξ such that if D denotes the domain bounded by $\pi \cup \xi$ and $f(\pi \cup \xi)$, then $f^i(D) \cap f^j(D) = 0$, whenever i and j are distinct integers, and $\bigcup_{-\infty < i < +\infty} f^i(\bar{D}) = S^n$. As a corollary to this theorem the author observes that if f is a quasi-translation of E^n then there is an unbounded polyhedron π such that if D denotes the domain bounded by π and $f(\pi)$, then $f^i(D) \cap f^j(D) = 0$, whenever i and j are distinct integers, and $\bigcup_{-\infty < i < +\infty} f^i(\bar{D}) = E^n$. This, in turn, implies the Kerékjártó-Sperner result, cited above, for quasi-translations on E^2 and it is the author's hope that it will be useful in a proof of the corresponding theorem for $n > 2$. *W. R. Utz.*

Plunkett, R. L. A theorem about mappings of a topological group into the circle. Michigan Math. J. **2**, 123–125 (1954).

Let G be a compact connected abelian group. According to Steenrod [Amer. J. Math. **58**, 661–701 (1936)] $H^1(G, \text{integers}) \approx \text{Char } G$. Let $B(G)$ be the group of continuous functions on G to the unit circle, modulo the null-homotopic ones. According to Dowker [ibid. **69**, 200–242 (1947); these Rev. **8**, 594] $H^1(G, \text{integers}) \approx B(G)$. The author shows that $B(G) \approx \text{Char } G$ when G is also metric. [See also K. Kodaira and M. Abe, Proc. Imp. Acad. Tokyo **16**, 167–172 (1940); these Rev. **2**, 5.] *A. D. Wallace* (New Orleans, La.).

Plunkett, Robert, L. Some implications of semi-1-connectedness. Proc. Amer. Math. Soc. **5**, 665–670 (1954).

A space X is semi-1-connected [see Wilder, Duke Math. J. **1**, 543–555 (1935), p. 549] at $x \in X$ if x has a neighborhood all of whose compact 1-cycles bound on X. Certain implications of this property are found for separable metric spaces; for example, if X is a locally connected continuum which is semi-1-connected at a non-local separating point x, then X is locally peripherally connected at x. And if T is a compact, totally disconnected set of non-local separating points of a locally connected continuum X, and X is semi-1-connected at each point of T, then for arbitrary $\epsilon > 0$, there exists a finite covering of T by disjoint open sets Y_i, such that for each i, diam $(Y_i) < \epsilon$ and $F(Y_i)$ is connected. *R. L. Wilder* (Ann Arbor, Mich.).

Ganea, Tudor. Contractilité des produits symétriques. Acad. Repub. Pop. Române. Stud. Cerc. Mat. **4**, 23–28 (1953). (Romanian. Russian and French summaries)

If X is metrizable, compact (locally compact) and contractible (locally contractible) then its nth symmetric potency [cf. Borsuk and Ulam, Bull. Amer. Math. Soc. **37**, 875–882 (1931)] is contractible (locally contractible). *R. H. Fox* (Princeton, N. J.).

Kosiński, A. A topological characterization of 2-polytopes. Bull. Acad. Polon. Sci. Cl. III. **2**, 321–323 (1954).

For any topological space X, denote by $\text{reg}_n X$ the set of its points having a neighborhood homeomorphic to Euclidean n-space. The main assertion: A finite 2-polytope X is completely characterized by the following three properties: (a) X is a compact metric ANR, (b) $X = A \cup B$, where $A \subset \text{reg}_2 X$ and B is a graph, and (c) almost all points of X have arbitrarily small neighborhoods U such that for every $x \in U \cap B$, boundary (U) is a deformation retract of $U - x$. No proof is given. An application shows that a compact metric space which is locally a 2-polytope is a 2-polytope. *J. Dugundji* (Los Angeles, Calif.).

Kosiński, A. On 2-dimensional topological divisors of polytopes. Bull. Acad. Polon. Sci. Cl. III. **2**, 325–328 (1954).

The results of the paper reviewed above are used to prove: If P is a finite polytope, if $P=A\times B$, and if $\dim A\leqq 2$, then A is also a polytope. This partially answers a question of Borsuk [Fund. Math. **31**, 137–148 (1938)]. Consequences: (a) If any finite polytope is fibered so that the dimension of the base (or fiber) is $\leqq 2$, then the base (or fiber) is also a polytope. (b) If a connected finite polytope E satisfying $\dim(E-\mathrm{reg}_n E)\leqq n-3$ is fibered so that the fiber (or base) has dimension $\leqq 2$, then the fiber (or base) is a closed manifold. *J. Dugundji* (Los Angeles, Calif.).

Miyazaki, Hiroshi. An application of the method of acyclic models. Proc. Japan Acad. **30**, 9–13 (1954).

Let K be a semi-simplicial complex [Eilenberg and Zilber, Ann. of Math. (2) **51**, 499–513 (1950); these Rev. **11**, 734], and $P(K)$ its associated semi-simplicial polytope [Hu, Pacific J. Math. **1**, 583–602 (1951); these Rev. **13**, 676]. Then, by a natural simplicial map f, K can be imbedded as a subcomplex of the singular complex of the topological space $P(K)$. In the present paper, the author gives a proof of the well-believed theorem that f induces an equivalence between the homology theory of the complex K and the singular homology theory of the topological space $P(K)$. The proof is an application of the method of acyclic models introduced by Eilenberg and MacLane [Amer. J. Math. **75**, 189–199 (1953); these Rev. **14**, 670]. *S. T. Hu.*

Granas, A. On local disconnection of Euclidean spaces. Fund. Math. **41**, 42–48 (1954).

Let F denote a closed subset of the n-sphere S_n. Let $a\in F$ and for $\epsilon>0$ let U_a^ϵ denote the ϵ-neighborhood of a in S_n. For $\epsilon>\eta>0$ let $b_0^{\epsilon,\eta}$ denote the number of components of $U_a^\epsilon-F$ which meet U_a^η. Let

$$b_0^\epsilon=\lim_{\eta\to 0} b_0^{\epsilon,\eta} \quad\text{and}\quad b_0(a, S_n-F)=\lim_{\epsilon\to 0} b_0^\epsilon.$$

The author proves that $b_0(a, S_n-F)$ is a topological invariant by showing that it can be determined from a local cohomotopy number, which is an invariant by the way it's defined. *E. Spanier* (Chicago, Ill.).

Olum, Paul. Homotopy type and singular homotopy type. Ann. of Math. (2) **60**, 317–325 (1954).

Let $S(X)$ be the singular complex of the arcwise connected space X, and $S^q(X)$ its q-skeleton. The author defines two spaces X, Y to be of the same singular n-homotopy type if there are singular maps $\varphi\colon S^{n+1}(X)\to S^{n+1}(Y)$, $\psi\colon S^{n+1}(Y)\to S^{n+1}(X)$ which, when restricted to the n-skeletons, show $S^n(X)$, $S^n(Y)$ belonging to the same homotopy type. If X, Y are CW-complexes, this notion is shown to coincide with the analogous one defined directly in terms of the complexes themselves. The author demonstrates that in case $\pi_1(X)\approx\pi_1(Y)$ the singular n-homotopy type can be determined by considering, instead of the full singular complex, any connected sub-complexes $M(X)$, $M(Y)$ satisfying

$$H^k(S(X), M(X); G)=H^k(S(Y), M(Y); G)=0$$

for all $k\leqq n+1$ and all systems of local groups G in X. Taking $M(X)$, $M(Y)$ to be minimal complexes, he proves: X, Y have the same singular n-homotopy type if and only if there are singular maps $\varphi\colon M^{n+1}(X)\to M^{n+1}(Y)$, $\psi\colon M^{n+1}(Y)\to M^{n+1}(X)$ with $\varphi'=\varphi|M^n(X)$, $\psi'=\psi|M^n(Y)$ satisfying $\varphi'\psi'=\psi'\varphi'=$identity. Applications are given. *J. Dugundji* (Los Angeles, Calif.).

Griffiths, H. B. A contribution to the theory of manifolds. Michigan Math. J. **2**, 61–89 (1954).

"Manifolds" are locally compact, metric, topological spaces for which certain local invariants [defined by the author in Proc. London Math. Soc. (3) **3**, 350–367 (1953); these Rev. **15**, 457] agree with these same invariants for locally euclidean spaces. Thus $\mathfrak{M}(\mathfrak{C}_\infty)$ uses homotopy invariants, while $\mathfrak{M}^n(\mathfrak{C}_V, G)$ and $\mathfrak{M}^n(\mathfrak{C}_C, G)$ use homology with Vietoris and Čech cycles respectively and with coefficients in various groups G. The author proves: (1) every $\mathfrak{M}^n(\mathfrak{C}_\infty)$ is an $\mathfrak{M}^n(\mathfrak{C}_V, I)$, but not conversely [$I$=the group of integers]; (2) every $\mathfrak{M}^n(\mathfrak{C}_V, I)$ is an $\mathfrak{M}^n(\mathfrak{C}_V, Q)$, Q being the field of rationals, and has the same global integer and rational Betti numbers in each dimension, but not conversely; (3) if G is discrete, every $\mathfrak{M}^n(\mathfrak{C}_V, G)$ is an $\mathfrak{M}^n(\mathfrak{C}_C, G)$ and conversely; (4) if G is a field, every $\mathfrak{M}^n(\mathfrak{C}_C, G)$ is a "generalized manifold" in the sense of Wilder, and conversely; (5) if $\mathfrak{M}^n(\mathfrak{C}_V, I)$ is compact and orientable, its global integer Betti numbers satisfy a Poincaré duality. *S. S. Cairns* (Strasburg).

Hirzebruch, Friedrich. On Steenrod's reduced powers, the index of inertia, and the Todd genus. Proc. Nat. Acad. Sci. U. S. A. **39**, 951–956 (1953).

M sei eine kompakte differenzierbare Mannigfaltigkeit der Dimension m. Es werden ohne Beweis wichtige explizite Beziehungen angekündigt (a) zwischen den Steenrodschen Quadraten Sq^i in M und den Stiefel-Whitney-Klassen von M; (b) wenn M orientiert ist, zwischen den Steenrodschen reduzierten Potenzen $\mathcal{P}_q^r$ in M und den Pontrjaginschen Klassen von M; (c) wenn $m=4k$ und M orientiert ist, zwischen dem Index $\tau(M)$ und den Pontrjaginschen Zahlen von M; (d) wenn $m=2n$ und M fast-komplex ist, zwischen den Sq^i, den $\mathcal{P}_q^r$ und τ einerseits und den Chernschen Klassen von M andererseits. Ferner wird für eine fast-komplexe Mannigfaltigkeit M das "Toddsche Geschlecht" definiert und untersucht. In den sehr inhaltsreichen und allgemeinen Ergebnissen sind als Spezialfälle Sätze von Thom, Wu und Todd enthalten.

Alle diese Beziehungen werden mit Hilfe "multiplikativer Polynomfolgen" formuliert. Eine Folge von Polynomen $K_j(c_1, \cdots, c_j)$, $j=1, 2, \cdots$, vom Gewicht j in den Unbestimmten $c_1, c_2, \cdots$ mit Koeffizienten aus einem Körper Γ, heisst multiplikativ, wenn die Potenzreihe

$$K=1+\sum K_j(c_1, \cdots, c_j)x^j$$

als Funktion von $C=1+\sum c_j x^j$ multiplikativ homomorph ist (d.h. $K(CC_1)=K(C)K(C_1)$). Zu jeder Potenzreihe $Q(x)=1+\sum\gamma_j x^j$, $\gamma_j\in\Gamma$, gehört eine wohlbestimmte multiplikative Folge K mit $K(1+x)=Q(x)$. Γ sei der Körper der rationalen Zahlen. Die zu $Q(x)=-x(e^{-x}-1)^{-1}$ gehörige Folge wird mit T_j, die zu $Q(x)=x^{\frac{1}{2}}(\mathrm{tgh}\, x^{\frac{1}{2}})$ gehörige mit L_j bezeichnet; $q^r T_{(q-1)r}$ ist ganz (mod q) für jede Primzahl q, ebenso $q^r L_{\frac{1}{2}(q-1)r}$ für Primzahlen $\geqq 3$. Die Resultate lauten nun so: (a) In der Mannigfaltigkeit M sei $U^i\in H^i(M, Z_2)$ definiert durch $\mathrm{Sq}^i v=U^i v$ für alle $v\in H^{m-i}(M, Z_2)$. Es ist $U^i=2^i T_i(w_1, \cdots, w_i)$ mod 2, wo $w_i\in H^i(M, Z_2)$ die Stiefel-Whitney-Klassen sind. (b) In M (orientiert) sei

$$s_q{}^r\in H^{2(q-1)r}(M, Z_q),$$

q=Primzahl$\geqq 3$, definiert durch $\mathcal{P}_q^r u=s_q{}^r u$ für alle

$$u\in H^{m-2(q-1)r}(M, Z_q).$$

Es ist $s_q{}^r=q^r L_{\frac{1}{2}(q-1)r}(p_1, \cdots, p_{\frac{1}{2}(q-1)r})$ mod q, wo $p_i\in H^{4i}(M, Z)$ die Pontrjaginschen Klassen von M sind. (c) Es sei $m=4k$, M orientiert, und $H^m(M, \Gamma)$ sei identifiziert mit Γ. Es

ist $\tau(M)=L_k(p_1,\cdots,p_k)$. (d) Ist $m=2n$ und M fast-komplex, so gilt $s_q{}^r=q^rT_{(q-1)r}(c_1,\cdots,c_{(q-1)r})$ mod q, ferner $U^{2i}=2^iT_i(c_1,\cdots,c_i)$ mod 2, und $U^{2i+1}=0$; dabei sind die $c_i\in H^{2i}(M,Z)$ die Chernschen Klassen von M; der Index $\tau(M)$ wird in den c_i durch die zu $Q(x)=x(\mathrm{tgh}\,x)^{-1}$ gehörigen Polynome ausgedrückt, auf Grund des bekannten Zusammenhangs zwischen den p_i und den c_i. Das Toddsche Geschlecht $T(M)$ einer fast-komplexen Mannigfaltigkeit der Dimension $2n$ wird definiert durch $T_n(c_1,\cdots,c_n)\in H^{2n}(M,\Gamma)$, als rationale Zahl betrachtet; $2^{n-1}T(M)$ ist eine ganze Zahl. (Bemerkung: Es ist vom Verfasser seither gezeigt worden, dass in nicht-singulären algebraischen Mannigfaltigkeiten $T(M)$ gleich dem arithmetischen Geschlecht ist.)

Verschiedene numerische Folgerungen aus den genannten Beziehungen sind erwähnt, ferner Eigenschaften des Index $\tau(M)$ und des Geschlechts $T(M)$, die sich aus formalen Relationen zwischen den Polynomen L_j bzw. K_j ergeben; sie betreffen insbesondere das virtuelle Geschlecht $T(x)$ für ein Element $x\in H^2(M,Z)$. *B. Eckmann* (Zürich).

Homma, Tatsuo, and Kinoshita, Shin'ichi. On a topological characterization of the dilatation in E^3. Osaka Math. J. **6**, 135–143 (1954).

Let h be any orientation-preserving autohomeomorphism of 3-dimensional Euclidean space E^3 such that, for any point p of E^3, the sequences $\{h^n(p)\}$ and $\{h^{-n}(p)\}$ converge to the origin and the point at ∞ respectively. Then h is topologically equivalent to the "dilation" d: $p\to\frac{1}{2}p$, that is to say $fhf^{-1}=d$ for some autohomeomorphism f of E^3. The proof proceeds in three stages. First it is established by general arguments that there is a polyhedral surface M that is separated from the origin by $h(M)$. Next, using a combinatorial lemma first proved by the reviewer [Ann. of Math. (2) **49**, 462–470 (1948); these Rev. **10**, 138] and an approximation theorem of Moise [ibid. **55**, 215–222 (1952); these Rev. **13**, 765], it is shown that a properly chosen sequence of cuts reduces M to a 2-sphere S that is separated from the origin by $h(S)$. Finally, using another approximation theorem of Moise [ibid. **56**, 96–114 (1952); these Rev. **14**, 72] and the classic Alexander theorem about polyhedral 2-spheres [Moise, ibid. **55**, 172–176 (1952); these Rev. **13**, 574], the Tietze deformation theorem is made to yield the transformation f required. *R. H. Fox.*

Homma, Tatsuo. On the existence of unknotted polygons on 2-manifolds in E^3. Osaka Math. J. **6**, 129–134 (1954).

The author considers a closed surface M of genus $p>0$ polyhedrally imbedded in 3-space E. His main result—that there must be in E a disc D whose interior lies in $E-M$ and whose boundary curve lies on M and does not bound any disc on M—is a special case of a theorem proved by the reviewer [Ann. of Math. (2) **49**, 462–470 (1948); these Rev. **10**, 138]. From this result the author derives, by an easy argument, a theorem which may be stated as follows: There are p discs $D_1,\cdots,D_p$ in E whose boundary curves $P_1,\cdots,P_p$ are pairwise disjoint, lie on M and reduce it to a sphere with $2p$ holes. (An additional argument would show that the discs $D_1,\cdots,D_p$ may be so selected that they are themselves pairwise disjoint.) Finally the author gives an example (a very pretty picture) of a surface M of genus 2 that is separated by every simple closed polygon on it that bounds a disc in $E-M$. (To the reviewer it appears that M may even be separated by every simple closed polygon that bounds a disc in E.) *R. H. Fox.*

Barcus, William. Note on cross-sections over CW-complexes. Quart. J. Math., Oxford Ser. (2) **5**, 150–160 (1954).

In the standard development of the theory of obstructions to cross-sections of fibre spaces and characteristic cohomology classes [see N. E. Steenrod, The topology of fibre bundles, Princeton, 1951, part III; these Rev. **12**, 522] it is customary to make the following assumptions. (a) The fundamental group of the fibre operates trivially on its higher homotopy groups. (b) The base space is a cell complex having the property that each closed cell is contractible. (c) The fibre space is locally a product space. In the present note the author gives a straightforward development of the theory of obstructions to cross-sections without these restrictive assumptions. He develops the theory for the case of fibre spaces in the sense of Serre [Ann. of Math. (2) **54**, 425–505 (1951); these Rev. **13**, 574] for which the base space is an arbitrary CW-complex in the sense of J. H. C. Whitehead [Bull. Amer. Math. Soc. **55**, 213–245 (1949); these Rev. **11**, 48] and the fibre is arcwise connected. The obstructions turn out to be elements of equivariant cohomology groups. Naturally, the case of 1- or 2-dimensional obstructions requires separate consideration under these very general conditions. *W. S. Massey.*

MacLane, Saunders. The homology products in $K(\Pi,n)$. Proc. Amer. Math. Soc. **5**, 642–651 (1954).

Cette question donne à l'auteur l'occasion de faire une revue de tous les "produits" utilisés en homologie et cohomologie singulière (simpliciale et cubique).

Soient $S(X)$ le complexe singulier (simplicial) d'un espace X, et $S^N(X)$ le complexe normalisé, obtenu en annulant les simplexes "dégénérés"; l'application $S(X)\to S^N(X)$ est une chaîne-équivalence. Soient $Q(X)$ le complexe singulier cubique de X, et $Q^N(X)$ le complexe normalisé, obtenu en annulant les cubes "dégénérés"; l'application $Q(X)\to Q^N(X)$ n'est pas une chaîne-équivalence. Mais l'homologie $H(Q^N(X))$ est canoniquement isomorphe à $H(S^N(X))\approx H(S(X))$; d'une façon précise, on connaît explicitement deux chaînes-équivalences naturelles

$$h\colon\ S^N(X)\to Q^N(X),\qquad k\colon\ Q^N(X)\to S^N(X)$$

avec $kh=$ identité. (Note: l'application k n'est pas mentionnée dans l'article.) Toute autre application naturelle h': $S^N(X)\to Q^N(X)$ est homotope à h; de même pour k.

On connaît [Eilenberg et Zilber, Amer. J. Math. **75**, 200–204 (1953); ces Rev. **14**, 670] une chaîne-équivalence naturelle

$$f\colon\ S(X_1\times X_2)\to S(X_1)\otimes S(X_2)$$

qui définit des produits (externes) dans les cochaînes simpliciales (formules d'Alexander-Whitney). De même, une chaîne-équivalence naturelle

$$g\colon\ Q^N(X_1\times X_2)\to Q^N(X_1)\otimes Q^N(X_2)$$

est explicitée; elle définit des produits dans les cochaînes cubiques, déjà utilisés par Serre. Le diagramme

$$\begin{array}{ccc} S^N(X_1\times X_2) & \xrightarrow{f^N} & S^N(X_1)\otimes S^N(X_2) \\ \downarrow h & & \downarrow h_1\otimes h_2 \\ Q^N(X_1\times X_2) & \xrightarrow{g} & Q^N(X_1)\otimes Q^N(X_2) \end{array}$$

est commutatif (exactement: l'auteur dit seulement qu'il est commutatif à une homotopie près; cf. démonstration du th. 6).

On définit des "produits" (externes) dans les chaînes singulières simpliciales, au moyen d'une chaîne-équivalence naturelle σ: $S(X_1)\otimes S(X_2)\to S(X_1\times X_2)$. Une telle σ a été explicitée par Eilenberg et MacLane [produit Δ, Ann. of Math. (2) **58**, 55–106 (1953); ces Rev. **15**, 54] à l'aide des "shuffles". De même, on explicite une chaîne-équivalence naturelle ρ: $Q^N(X_1)\otimes Q^N(X_2)\to Q^N(X_1\times X_2)$. (Note: $f^N\sigma^N$ est l'application identique de $S^N(X_1)\otimes S^N(X_2)$, et $g\rho$ l'application identique de $Q^N(X_1)\otimes Q^N(X_2)$.) Le diagramme

$$\begin{array}{ccc} Q^N(X_1)\otimes Q^N(X_2) & \xrightarrow{\rho} & Q^N(X_1\times X_2) \\ \downarrow k_1\otimes k_2 & & \downarrow k \\ S^N(X_1)\otimes S^N(X_2) & \xrightarrow{\sigma^N} & S^N(X_1\times X_2) \end{array}$$

est commutatif. (Note: l'auteur indique seulement le diagramme analogue, qui utilise h et $h_1\otimes h_2$, et n'est commutatif qu'à une homotopie près; cf th. 1.) Lorsque X est muni d'une application $X\times X\to X$, σ et ρ définissent des produits internes dans l'homologie singulière $H(S(X))\approx H(Q^N(X))$. Toute autre application naturelle σ' (resp. ρ') définit la même multiplication dans l'homologie.

L'auteur applique ces résultats à la multiplication dans $H(\Pi, n)$, homologie du complexe minimal $K(\Pi, n)$ de n'importe quel espace X de type $\mathfrak{K}(\Pi, n)$, i.e. tel que $\pi_i(X)=0$ pour $i\neq n$, $\pi_n(X)=\Pi$. Supposons que X soit un H-espace (c'est possible: prendre pour X l'espace des lacets d'un Y de type $\mathfrak{K}(\Pi, n+1)$). Soit $S_n(X)$ le sous-complexe de $S(X)$ engendré par les simplexes dont le $(n-1)$-squelette est au point-base e, élément unité de la multiplication de X. Les applications canoniques $S(X)\leftarrow S_n(X)\to K(\Pi, n)$ sont des chaînes-équivalences, compatibles avec la structure multiplicative de $S(X)$ et $S_n(X)$ (déduite de σ) et celle de $K(\Pi, n)$ (définie par Eilenberg-MacLane, loc. cit., à l'aide des "shuffles"). Donc l'isomorphisme canonique $H(X)\approx H(\Pi, n)$ est compatible avec les produits internes; la multiplication de $H(X)$ est associative et anticommutative, comme celle de $K(\Pi, n)$. *H. Cartan* (Paris).

GEOMETRY

Spezia, Clotilda. Harmonic points and loci connected with the Frégier theorem. Math. Mag. **28**, 13–19 (1954).

Manara, C. F. L'aspetto algebrico di un fondamentale teorema di geometria descrittiva. Period. Mat. (4) **32**, 142–149 (1954).

***Charrueau, André. Complexes linéaires. Faisceaux de complexes linéaires. Suites et cycles de complexes linéaires conjugués.** Mémoir. Sci. Math., no. 120. Gauthier-Villars, Paris, 1952. 83 pp. 900 francs.

Le livre contient les propriétés métriques et projectives des complexes linéaires et des faisceaux de ces complexes. La matière s'ajoute à celle du livre de Koenigs "La géométrie réglée" [Gauthier-Villars, Paris, 1895] mais elle est accomplie et approfondie. La théorie traitée fut publié par l'auteur dans C. R. Acad. Sci. Paris en 1948–1952.

Le travail a trois parties: La première partie contient les propriétés métriques et projectives des complexes linéaires bien connues. On y ajoute quelques remarques concernant les courbes d'un complexe linéaire, la transformation par rapport à un complexe linéaire et enfin les remarques sur les surfaces polaires réciproques par rapport à un complexe linéaire. Dans la seconde partie du travail on considère les faisceaux de complexes linéaires et en particulier les suites et cycles des complexes linéaires conjugués. Ceux-ici sont les ensembles des complexes linéaires qu'on peut obtenir en utilisant la transformation par polaires réciproques par rapport au complexe linéaire d'un faisceau: Soient C_1 et C_2 deux complexes d'un faisceau qui sont différents et non spécials. Cherchons le complexe linéaires C_3 conjugué de C_1 par rapport à C_2 et le complexe C_4 conjugué de C_2 par rapport à C_3, etc. Dans la suite S (cycle C) le complexe C_{j+2} est conjugué de C_j par rapport à C_{j+1} quelque soit j. Dans un cycle C de p complexes linéaires conjugués en outre C_1 est conjugué de C_{p-1} par rapport à C_p et C_2 le conjugué de C_p par rapport à C_1. La suite (cycle) appartient à un faisceau.

Les propriétés des transformations par polaires réciproques relatives à des complexes d'un faisceau et les produits de ces transformations sont aussi étudiés. La troisième partie concerne les remarques à la théorie métrique des cas particuliers des faisceaux de complexes linéaires, par exemple, le cas où les paramètres des complexes d'un faisceau sont égaux. *F. Vyčichlo* (Prague).

Morduhaĭ-Boltovskoĭ, D. D. On the arc of a curve of second degree in the Lobačevskiĭ plane. Doklady Akad. Nauk SSSR (N.S.) **95**, 449–450 (1954). (Russian)

It is shown that the rectification of ellipses and hyperbolas in the hyperbolic plane leads to hyperelliptic integrals of the first and second kind. *H. Busemann* (Göttingen).

Antonescu, M. Sur un théorème de Hilbert dans la géométrie non euclidienne de Lobatchewski-Bolyai. Acad. Repub. Pop. Române. Stud. Cerc. Mat. **4**, 197–211 (1953). (Romanian. Russian and French summaries)

Hilbert's theorem on the non-realizability of the hyperbolic plane in E^3 is proved again, also by considering the Tchebychev nets formed by the asymptotic lines. General Tchebychev nets in the hyperbolic plane are studied first, and it is shown that the lines in such a net always possess an envelope, whereas the asymptotic line on a surface of constant negative curvature without singularities cannot have an envelope. *H. Busemann* (Göttingen).

Wright, Jesse. Quasi-projective geometry of two dimensions. Michigan Math. J. **2**, 115–122 (1954).

Als quasi-projective Geometrie bezeichnet Verf. eine Geometrie, deren Kleinsche Automorphismengruppe die um die Menge der Correlationen erweiterte Gruppe der Collineation ist. Die quasi-projective Geometrie entsteht aus der projectiven Geometrie durch Eliminierung des Begriffes Punkt. Zur axiomatischen Begründung einer ebenen quasi-projectiven Geometrie stellt Verf. 5 Axiome auf, in die eine Sorte von Elementen $x, y, u, \cdots$ und eine Incidenzrelation I zwischen ihnen eingeht: die Incidenzrelation ist irreflexiv; jedes Element incidiert mit mindestens 3 weiteren Elementen; zu jedem Tripel verschiedener Elemente gibt es ein Element, das mit mindestens 2 Elementen des Tripels incidiert; wenn in einer Kette aus 4 verschiedenen Elementen jedes Element mit dem nächsten incidiert, so incidiert das letzte nicht mit dem ersten; das entsprechende gilt für eine Kette aus 5 verschiedenen Elementen. Jedes dieser Axiome ist von den Übrigen unabhängig. Die projective Ebene, in der zwischen Punkten und Geraden nicht unterschieden wird, aber die gewöhnlichen Incidenzbeziehungen beibehalten werden, liefert ein Modell für diese quasi-projective Geometrie. Weiter werden die Relationen D und C eingeführt: xDy besagt, dass es ein u gibt, so dass x mit u und u

mit y incidiert; x und y heissen dann von gleicher Dimension. xCy besagt, dass xDy nicht gilt und x mit y nicht incidiert; x und y heissen dann complementär. Die Relation D bewirkt eine Einteilung der Elemente in zwei Klassen, die den Punkten und Geraden der projectiven Geometrie entsprechen, die sich aber in der quasi-projectiven Geometrie nicht getrennt definieren lassen. Verf. beweist einige Sätze über die Relationen I, D, C. Die vorliegende Note erstrebt noch keine vollständige Theorie der quasi-projectiven Geometrien. *R. Moufang* (Frankfurt am Main).

Convex Domains, Extremal Problems, Integral Geometry

***Pogorelov, A. V. Odnoznačnaya opredelennost' obščih vypuklyh poverhnostei. [Unique determination of general convex surfaces.]** Monografii Instituta Matematiki, vyp. II. Akad. Nauk Ukrainskoĭ SSR, Kiev, 1952. 69 pp.

The book contains a proof of the general rigidity theorem for closed convex surfaces announced previously [Doklady Akad. Nauk SSSR (N.S.) **79**, 739–742 (1951); these Rev. **13**, 271]. The theorem is this: If for two closed convex surfaces in E^3 a one-to-one mapping exists such that corresponding curves have equal lengths, then the surfaces are congruent, i.e., one can be transformed into the other by a (not necessarily orientation-preserving) motion of E^3.

Contrary to the impression which the reviewer obtained from the brief earlier note, the proof is long and involved, but very appealing to those who like subtle, often purely geometric arguments. An introduction deals with the history of the problem and the significance of the methods developed by A. D. Alexandrov and the author. Next comes a resumé of results from Alexandrov's theory. Chapter II contains, in the reviewer's opinion, the only new material which is likely to have applications outside the problem in question and, possibly, its extension to unbounded surfaces. Let γ be a curve in E^3; for each inscribed polygon P with angles $\alpha_1, \cdots, \alpha_n$, $0<\alpha_i<\pi$, at the vertices form $\omega(P)=\sum_{i=1}^{n}(\pi-\alpha_i)$. Then $\omega(\gamma)=\sup_P \omega(P)$ is called the variation of the curvature. If $\omega(\gamma)<\infty$ then γ is rectifiable, possesses everywhere one-sided tangents etc. The deepest property is: Let γ be any curve which possesses everywhere a right tangent. Take any given number of consecutive points $p_1, \cdots, p_n$ in γ and form the sum $\omega(p)=\omega(p_1, \cdots, p_n)$ of the angles between the right tangents at two consecutive p_i. Then $\omega(\gamma)=\sup_p \omega(p)$. The application to convex surfaces comes through the following non-trivial fact: If the curve γ lies on a piece of a convex surface whose supporting planes form with some fixed plane Q angles bounded away from $\pi/2$, and if the projection γ' of γ on Q has a finite $\omega(\gamma')$, then $\omega(\gamma)<\infty$.

The remainder is essentially one large indirect proof, i.e., consists of conclusions drawn from the existence of two non-congruent isometric closed convex surfaces. It is impossible to indicate the various steps here.

The author emphasizes that this general theorem in conjunction with other results of Alexandrov and the author can be used to solve important problems in the realm of classical differential geometry, but, at least hitherto, inaccessible to the classical methods. This is highly inconvenient for those who justify their disregard for the new methods by their exclusive interest in smooth surfaces. The author gives the following very illuminating example. Alexandrov's "gluing theorem" yields as a very special case: Let F and G be two smooth convex surfaces homeomorphic to a disk. If a correspondence between the boundary curves of F and G exists which preserves length and is such that the sum of the geodesic curvatures of the curves at corresponding points is non-negative, then $F \cup G$ (with boundary points identified by the correspondence) is realizable as a convex surface in E^3. Apply this to F as a spherical cap bounded by a circle C and G as a plane domain bounded by an ellipse E with the same length as C. The image F' of F under the realization of $F \cup G$ in E^3 is isometric, but not congruent to F. Alexandrov's result does not imply differentiability properties of F', but it follows from a result of the author that, as isometric image of an analytic surface, F' is itself analytic, and from the uniqueness theorem of the present book, that F' varies continuously if C changes continuously (through ellipses say) into E. Thus F can be bent continuously into F'. *H. Busemann* (Göttingen).

Pogorelev, A. V. The unique determination of infinite convex surfaces. Doklady Akad. Nauk SSSR (N.S.) **94**, 21–23 (1954). (Russian)

Let F be an unbounded complete convex surface in E^3 of class C^3 with integral curvature 2π. Then F is proved to be rigid by a method similar to that used by Herglotz in the bounded case. The author had previously proved this theorem by a different method and under the additional hypotheses that F lies inside some paraboloid [Trudy Mat. Inst. Steklov. **29** (1949); these Rev. **12**, 437].

H. Busemann (Göttingen).

***Alexandrov, A. D. Synthetic methods in the theory of surfaces.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 162–175. Edizioni Cremonese, Roma, 1954. 4000 Lire.

The methods developed by the author for convex surfaces and more generally for surfaces of bounded curvature, are briefly described, in particular the method of approximation by polyhedra and that of gluing. Their applicability to classical problems on the one hand, and the bridging of the gap between these methods and the classical approach by Pogorelov on the other, are discussed. There are also particular examples to illustrate the powers of the method of gluing. Some results to which the author refers have, to the reviewer's knowledge, hitherto not been known outside the USSR. *H. Busemann* (Göttingen).

Klee, V. L., Jr. Common secants for plane convex sets. Proc. Amer. Math. Soc. **5**, 639–641 (1954).

Dans un article [Atti 4° Congresso Un. Mat. Ital., Taormina, 1951, vol. II, Edizioni Cremonese, Roma, 1953, pp. 456–464; ces Rev. **15**, 150], le référent a été conduit à signaler qu'étant donné un ensemble (F) quelconque de corps convexes d'un plan, si toute droite parallèle à une direction donnée du plan coupe un corps (F) au plus, et s'il existe une droite au moins coupant les quatre éléments de tous les systèmes de quatre corps de (F), il existe une droite au moins coupant tous les corps de (F).

L'auteur montre que le résulatat précédent subsiste si l'on remplace le nombre quatre qui figure dans l'énoncé précédent par le nombre trois. La démonstration est basée sur la considération d'un ensemble P de points du plan ainsi défini: Pour tout système de trois corps de (F) on peut choisir trois points (non nécessairement distincts) appartenant chacun à l'un des trois corps envisagés. P est

l'ensemble formé par tous les triples de points ainsi obtenus. L'auteur envisage alors les différents produits $F \cap P$. Il attache à chacun d'eux le plus petit polygone convexe G le contenant, ce qui donne un ensemble de polygones convexes (G), et il démontre la proposition pour cet ensemble de polygones (G), ce qui entraîne la même proposition pour l'ensemble initial. *P. Vincensini* (Marseille).

Levi, F. W. Ein geometrisches Überdeckungsproblem. Arch. Math. **5**, 476–478 (1954).

Let $C=C(1)$ denote a convex region in the Euclidean plane, $C(\rho)$ a similar region decreased in the linear ratio $1:\rho$ $(\rho>1)$. Let $S(C, \rho)$ denote the smallest number of regions $C(\rho)$ that will cover C, and $S(\rho)$ the maximum $S(C, \rho)$ for various shapes C. The author proves that $S(1)=3$ and that, as ρ increases, $S(\rho)$ is asymptotically $\alpha\rho^2$, where $1<\alpha<3/2$. *H. S. M. Coxeter* (Toronto, Ont.).

Federer, Herbert. Some integralgeometric theorems. Trans. Amer. Math. Soc. **77**, 238–261 (1954).

The two integral-geometric theorems of the paper concern a pair of analytic sets A, B in Euclidean n-space, which are respectively of finite k- and l-dimensional measure, where $k+l>n$ (equality permitted in one theorem). B is assumed to be on a Lipschitzian l-variety and A to be on a countable sum of Lipschitzian k-varieties. It is shown that for almost all isometries f of n-space, the intersection E_f of A and $f(B)$ lies on a countable sum of Lipschitzian $(k+l-n)$-varieties and that the appropriate multiple of the Haar integral in f of the $(k+l-n)$-measure of E_f is the product of the k- and l-measures of A and B. *L. C. Young.*

Algebraic Geometry

Brusotti, Luigi. Fasci reali di curve algebriche. Rend. Sem. Mat. Fis. Milano **23** (1952), 21–35 (1953).

Nei riguardi dei fasci reali di curve algebriche nel piano e sulla superficie algebrica reale, si illustrano metodi e si riassumono risultati in modo inteso a lumeggiare l'indole di un campo di studi non molto coltivato. (Author's summary.) *H. T. Muhly* (Iowa City, Iowa).

Severi, Francesco. Geometria algebrica ed algebra astratta. Rend. Sem. Mat. Fis. Milano **23** (1952), 114–151 (1953).

An expository article in which the author surveys the most significant applications of abstract algebra to modern algebraic geometry. He stresses the importance of the arithmetic trend in algebraic geometry and points out the historical origins of this trend and the connections with the work of Italian geometers. The following is the author's summary of the article: Primi concetti d'algebra astratta. Divisibilità degli ideali. Caratteristica ed estensione d'un corpo. Anelli noetheriani e ideali di polinomi. Le varietà algebriche. Punti generici, generali, qualunque. Il principio della conservazione del numero. La forma associata ad una varietà algebrica. Valutazioni. Varietà localmente e aritmeticamente normali. Conclusione. *O. Zariski.*

***Severi, Francesco. Sulla caratterizzazione dei corpi di funzioni quasi abeliane.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 21–26. Edizioni Cremonese, Roma, 1954. 4000 Lire.

Viene dimostrata l'equivalenza delle seguenti definizioni a), b) dei corpi di funzioni quasi abeliane di π variabili, già in precedenza ottenuta dall'autore con una ipotesi restrittiva [Pont. Acad. Sci. Scripta Varia, v. 4, 1947; questi Rev. **9**, 578]. a) Corpo delle funzioni razionali sopra una varietà algebrica V_π (di dimensione π) possedente un gruppo continuo abeliano ∞^π di trasformazioni birazionali generalmente transitivo; b) corpo delle funzioni razionali sul prodotto di una varietà abeliana di Picard W_p e di uno spazio lineare S_δ $(p+\delta=\pi)$. L'autore rileva che il risultato, da lui ottenuto per via algebrico-geometrica, ha un fondamento topologico-differenziale in quanto equivale alla determinazione di una opportuna struttura fibrata sulla V_π, previa una conveniente trasformazione birazionale della varietà. *E. Martinelli* (Genova).

Matsusaka, Teruhisa. A remark on my paper "Some theorems on abelian varieties." Nat. Sci. Rep. Ochanomizu Univ. **4**, 172–174 (1954).

L'auteur corrige et simplifie la démonstration du théorème suivant [cf. mêmes Rep. **4**, 22–35 (1953); ces Rev. **15**, 983]: si une variété V à loi de composition normale définie sur k est birationnellement équivalente à une variété abélienne, elle est birationnellement équivalente sur k à une variété abélienne projective. *P. Samuel* (Clermont-Ferrand).

Matsusaka, Teruhisa. On the theorem of Castelnuovo-Enriques. Nat. Sci. Rep. Ochanomizu Univ. **4**, 164–171 (1954).

Soient V^n $(n\geqq 3)$ une variété projective sans singularités de dimension $n-2$, et (S) un système linéaire de diviseurs sur V, contenant des variétés W^{n-1} sans singularités de dimension $n-3$. Un critère d'équivalence de première espèce [cf. Weil, Proc. Nat. Acad. Sci. U. S. A. **38**, 258–260 (1952); ces Rev. **13**, 867] montre que l'image canonique de la variété de Picard $P(V)$ de V dans $P(W)$ est isogène à $P(V)$. Si, de plus, on a $\dim(S)\geqq 2$, si le système caractéristique de (S) contient des variétés sans singularités en dimension $n-4$, et si W est un diviseur générique de (S), alors l'application d'un critère d'équivalence de seconde espèce (loc. cit.) à W et à une courbe générique de W montre que $P(V)$ est isogène à $P(W)$. Ceci montre que, si V^n $(n\geqq 3)$ est absolument normale et sans singularités en dimension $n-2$, et si W^{n-1} est une section hyperplane générique de V, alors la variété de Picard $P(V)$ est isomorphe à $P(W)$ en tant que groupe, et est purement inséparable sur $P(W)$; l'existence de systèmes linéaires complets sur V dont la trace sur W est complète montre alors que, en fait, $P(V)$ et $P(W)$ sont isomorphes (en tant que variétés abéliennes). Par dualité les variétés d'Albanese de V et W sont aussi isomorphes. Ceci généralise le théorème d'Enriquès-Castelnuovo disant que les premiers nombres de Betti (c.à.d. les doubles des irrégularités) de V et W sont égaux. *P. Samuel* (Clermont-Ferrand).

Barsotti, I. A note on abelian varieties. Rend. Circ. Mat. Palermo (2) **2** (1953), 236–257 (1954).

If an absolutely irreducible variety V, defined on an arbitrary field k, has a normal law of composition $(P, Q)\to PQ=R$ (P, Q, R being points of V), and a proper sub-variety W, such that the following properties hold: (a) if $\bar{k}$ is the algebraic closure of k, and P and $Q \in V_{\bar{k}}-W_{\bar{k}}$, then $PQ \in V_{\bar{k}}-W_{\bar{k}}$; (b) the set $V_{\bar{k}}-W_{\bar{k}}$ with the given law of composition is a group; then V is called a group variety and W its degeneration locus. When W is empty, V is called an abelian variety.

It is shown that if V is an abelian variety over k, there exists a variety A, defined over k, which is birationally equivalent to V and is an abelian variety (with the induced

law of composition), and has the property that each point of A (over $\bar{k}$) is simple. It is also shown that if V is an absolutely irreducible variety with a normal law of composition, there exists a group variety (with an induced law of composition) birationally equivalent to V, and a necessary and sufficient condition is given for g to be abelian.

W. V. D. Hodge (Cambridge, England).

Conforto, Fabio. Sulle trasformazioni in sè della varietà quasi abeliana di Picard, che sono rappresentate da congruenze lineari tra gli integrali virtualmente di prima specie. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **13**, 219–248 (1954).

Sia K un corpo di funzioni quasi abeliane di π variabili $u_1, u_2, \cdots, u_\pi$ i cui interi caratteristici valgano p, δ_1, δ_2, ρ ($\pi = p+\delta_1+\delta_2$, $\rho \leqq p$, $\rho \leqq \delta_2$), ω una matrice (a π righe e $\pi' = 2p+\delta_1 \leqq 2\pi$ colonne) di periodi (primitivi) relativa ad esso, V_π la varietà (quasi abeliana) di Picard associata a K. Nel presente lavoro l'autore si pone il problema della determinazione delle trasformazioni T di V_π in sè rappresentate da congruenze lineari tra gli integrali virtualmente di prima specie $u_1, \cdots, u_\pi$: $u' \equiv \Lambda u + \lambda \pmod{\omega}$, con Λ matrice quadrata non degenere d'ordine π e λ matrice a π righe ed una colonna. Tale ricerca equivale a quella di tutte le relazioni generalizzate di Hurwitz della ω del tipo $\Lambda\omega = \omega I$ con I matrice quadrata (d'ordine π') ad elementi interi ed unimodulare. Utilizzando i risultati contenuti in una precedente memoria [Ann. Mat. Pura Appl. (4) **28**, 299–315 (1949); questi Rev. **12**, 172] l'autore determina tutte le trasformazioni T per ω qualsiasi nel caso $\rho = p$, e per ω a moduli generali se $\rho < p$, caratterizzando le possibilità e le modalità della presenza di trasformazioni distinte da quelle di prima specie ($u' \equiv -u+\lambda$) e di seconda specie ($u' \equiv u+\lambda$). Le T costituiscono un gruppo G e si dividono in schiere ∞^π, classi laterali di G rispetto al sottogruppo invariante H (a π parametri) delle trasformazioni di seconda specie, e la completa conoscenza di G equivale a quella del "gruppo delle schiere" G/H. Nel caso in cui $\rho = p$ le T si distribuiscono in infinite schiere che se $\rho = \delta_2$ dipendono da parametri interi (eccetto il caso in cui $\rho = p = \delta_2 = 0$, $\delta_1 = 1$ in cui vi sono solo le trasformazioni di prima e di seconda specie), mentre se $\rho < \delta_2$ dipendono da parametri che variano in parte in modo discreto ed in parte in modo continuo. Qualora sia $\rho < p$ (ed ω generica) esistono su V_π solo trasformazioni di prima e di seconda specie se $\rho = \delta_2$, mentre se $\rho < \delta_2$ le T si distribuiscono ancora in infinite schiere.

D. Gallarati (Genova).

Benedicty, Mario. Le varietà $\omega M \omega_{-1} = 0$. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **13**, 89–98 (1954).

In a projective space S_{2p^2-1}, whose homogeneous coordinates are the elements of a $p \times 2p$ complex matrix ω, the author studies the algebraic varieties $V(M)$ defined by an equation $\omega M \omega_{-1} = 0$, $M = -M_{-1}$ being a $2p \times 2p$ complex matrix, and $|M| \neq 0$. Such a $V(M)$ is irreducible, rational, of normal dimension $\frac{1}{2}(3p^2+p-2)$, and is the complete and generally simple intersection of $a = \binom{p}{2}$ quadrics; its order is consequently 2^a. Moreover, the only multiple points of a $V(M)$ are the points for which ω has rank $\leqq p-2$. If M is with integer elements, $V(M)$ contains (as a proper subset) the points which are images of the Riemann matrices ω of genus p fulfilling the singular relation $\omega M \omega_{-1} = 0$. The knowledge of $V(M)$ makes easier the study of the $p \times 2p$ matrices which are limits of Riemann matrices of genus p; these are important in the theory of quasi-abelian functions.

M. Rosati (Rome).

Benedicty, Mario. La geometria algebrica astratta e i campi di funzioni algebriche. Archimede **6**, 131–139 (1954).

Expository paper.

Roth, Leonard. Sull'estensione di un teorema di Castelnuovo-Humbert. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **15** (1953), 376–380 (1954).

The author obtains a number of properties of a V_r ($r > 2$) which contains a congruence Γ, of index ν, of rational curves. He obtains a number of conditions under which the birationality or unirationality of V_r can be asserted (a complete statement is rather involved). He also proves that, on any V_r containing a congruence of rational curves, (i) the operation of successive adjunction terminates after a finite number of steps, so that the geometric genus and all the plurigenera of V_r vanish; (ii) if the congruence is birational, V_r is superficially regular; (iii) if a V_3 contains a rational congruence of rational curves, of index 1, then it is completely regular.

J. A. Todd (Cambridge, England).

Scott, D. B. Correspondences with unequal valencies. Proc. Cambridge Philos. Soc. **50**, 639–640 (1954).

It is pointed out that any hyperelliptic surface admits correspondences T with itself, such that T and T^{-1} have unequal valencies in the sense of Albanese and neither is a valency correspondence in Severi's sense.

H. T. Muhly.

Burniat, Pol. Surfaces algébriques à système canonique dégénéré. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16**, 209–214 (1954).

L'auteur donne ici des exemples de surfaces algébriques dont les courbes canoniques sont toutes décomposées. Le premier exemple vient d'un plan quadruple Abélian dont la courbe de diramation se compose d'une conique C^2, de quatre droites issues d'un point A et de $2r+2$ droites ($r > 2$) issues d'un point O. Si i des points simples communs à C^2 et aux droites issues de A appartiennent à autant des droites issues de O, alors on a pour le plan quadruple: $p_a = p_g = r$, $p^{(1)} = 8p_g - i + 1$, $P_2 = 9p_g - i + 1$; et ses courbes canoniques se composent d'une courbe fixe de genre $2p_g + 5 - i$, d'une courbe fixe de genre virtuel -1 et de courbes de genre 3 d'une faisceau linéaire. Analoguement, le plan quadruple Abélian ayant comme courbe de diramation: deux droites issues d'un point A, une courbe C^4 ayant A comme point double et un groupe de $2r$ droites ($r > 2$) issues d'un point O, dans l'hypothèse que i de ces dernières $2r$ droites passent chacune par un des 4 points simples communs aux deux premières lignes, donne lieu à: $p_a = p_g = r$, $p^{(1)} = 8p_g - 7 - i$, $P_2 = 9p_g - 7 - i$; et les courbes canoniques se composent d'une courbe fixe de genre virtuel $2p_g - 1 - i$ et de courbes de genre 3 d'un faisceau linéaire.

Sans démonstration, l'auteur énonce enfin que, si dans le premier exemple la conique donnée se décompose en deux droites, on obtient un plan quadruple irrégulier, avec $q = 2$, $p_g = r$, $p^{(1)} = 8p_g - 7$, $P_2 = 9p_g - 7$; les courbes canoniques se composent encore, dans leur partie variable, de courbes de genre 3 d'un faisceau linéaire. On en conclut que, pour toute valeur de p_g, il est des surfaces algébriques irréductibles, soin régulières que irrégulières, dont le système canonique, et dehors d'une partie fixe, se composent de courbes de genre 3 d'un faisceau linéaire.

E. G. Togliatti (Gênes).

*Hodge, W. V. D. **Differential geometry and the theory of algebraic varieties.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 13–20. Edizioni Cremonese, Roma, 1954. 4000 Lire.

In this lecture the author discusses the method of sheaves (stalks, faisceaux) as applied to algebraic varieties in the complex domain. The main concepts are briefly introduced and some of the preliminary results on exact sequences of sheaves are stated. One application is presented in full, and that is a sheave-theoretic proof of Lefshetz's theorem on the transcendental characterization of algebraic 2-cycles on an algebraic surface. This proof, due to Kodaira and Spencer, is presented with a variation due to the author.

O. Zariski (Cambridge, Mass.).

van der Waerden, B. L. Zur algebraischen Geometrie. 17. Lokale Dimension und Satz von Eckmann. Math. Ann. **128**, 128–134 (1954).

[For part 16 see Math. Ann. **125**, 314–324 (1953); these Rev. **14**, 1012.] Let $h_1, \cdots, h_r$ be polynomials in n letters with coefficients in an overfield of a field K; denote by $d(h_1, \cdots, h_r)$ the highest of the dimensions of the components containing the origin of the set defined by the equations $h_i=0$ (in the n-dimensional affine space over an universal domain containing K). The author proves that $d(h_1', \cdots, h_r') \geqq d(h_1, \cdots, h_r)$ if the coefficients of the polynomials h_i' result from those of the h_i's by a specialisation over K. As an application of this, it is proved that, if $r=n$, if $\sum_{i=1}^{n} x_i h_i = 0$ and if n is odd, then $d(h_1, \cdots, h_n) > 0$. This had been proved by Habicht [Comment. Math. Helv. **18**, 154–175 (1946); these Rev. **8**, 61] in the case where $h_1, \cdots, h_n$ are homogeneous; in the case where K is the field of complex numbers, it gives a special case of a topological theorem of Eckmann [ibid. **15**, 1–26 (1943); these Rev. **4**, 173].

C. Chevalley (New York, N. Y.).

van der Waerden, B. L. Zur algebraischen Geometrie. 18. Ketten in mehrfach-projektiven Räumen. Math. Ann. **128**, 135–137 (1954).

In part 16 [Math. Ann. **125**, 314–324 (1953); these Rev. **14**, 1012], the author made use of the properties of cycles in a product of projective spaces. In the present paper, he shows how it is possible to deduce these properties from the analogous ones for cycles in a projective space by imbedding the product of projective spaces into a projective space.

C. Chevalley (New York, N. Y.).

van der Waerden, B. L. Zur Konstruktion des Resultantensystems für homogene Gleichungen. Arch. Math. **5**, 371–375 (1954).

This paper contains a proof of the existence of a resultant-system of a system of algebraic equations, together with a plea for the use of elimination theory in algebraic geometry instead of the "geometric language" of A. Weil. The proof is simple and elegant. The plea is unconvincing. To quote the example given by the author, it seems to the reviewer that it is even more "handgreiflich und suggestiv" to say: "take the projection of the variety defined by the equations $F_\mu(\alpha, \beta)=0$ on the α-space" than to say "eliminate the β's from the equations $F_\mu(\alpha, \beta)=0$, so as to obtain equations in the α's only". Besides, the first directive has the advantage of pointing to a well defined object, viz. the projection of a subset of a product on one of the factors, while there are many systems of equations of which it can be said that they are obtained by elimination of the β's from the equations $F_\mu(\alpha, \beta)=0$.

C. Chevalley (New York, N. Y.).

Gröbner, Wolfgang. Sopra un teorema di Severi. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **11**, 217–223 (1952).

Let f be a homogeneous prime element in the ring of polynomials $R_n=k[x_0, \cdots, x_n]$. Suppose that f has one of the following properties. A) Every $(n-2)$-dimensional subvariety of f is the complete intersection of f and some g in R_n. B) The difference ring $R_n-(f)$ is a ring with unique factorization. C) The relation $ad-bc=gf$, where a, b, c, d, g are forms in R_n, implies that one of the rows (columns) of the latter determinant is a multiple mod (f) of the other row (column). Then it is shown that f has also the other two properties. Furthermore, the author suggests that by counting the numbers of arbitrary coefficients in $ad-bc=gf$ we may arrive at: A) is true if $n=3$ and f is general of degree $\geqq 4$ [M. Noether, Abh. Akad. Wiss. Berlin **1882** (1883)]; A) is true if $n=4$ and f has no singular points [F. Severi, Memorie scelte, v. 1, Zuffi, Bologna, 1950, pp. 397–404; these Rev. **12**, 383]. *F. J. Terpstra* (Bandung).

Zariski, Oscar. Applicazioni geometriche della teoria delle valutazioni. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **13**, 51–88 (1954).

In this article, which is mostly expository, the author begins by showing how a sequence of infinitely near points on a surface leads naturally to the notion of a place. He then discusses places and valuations of an algebraic function field (of arbitrary dimension), birational and algebraic correspondences, normal varieties and their behavior under birational transformations, the Riemann manifold of an algebraic function field, local uniformization.

The following theorem had previously been proved by the author only for surfaces but is here proved in general: Let K be an algebraic function field of dimension r over k, let $\{V_i\}$ be a sequence of projective models of K, let P_i be a point of V_i at which V_i is locally normal. Assume the birational transformation from V_{i+1} to V_i is regular at P_{i+1} and sends P_{i+1} into P_i. Then the union of the quotient rings $Q(P_i)$ is a valuation ring if and only if every place of K having center on V_i at P_i for all i is of first kind with respect to V_i for sufficiently large i. [For the case $r=2$, see Zariski, Ann. of Math. (2) **40**, 639–689 (1939); these Rev. **1**, 26.] These two equivalent statements will always be true if each V_i is non-singular and if each P_i has dimension $r-2$ over k.

I. S. Cohen (New York, N. Y.).

Differential Geometry

Pliś, A. Haantjes and Alt curvatures in abstract metric spaces. Ann. Polon. Math. **1**, 29–33 (1954).

The paper exhibits a metric transform of the segment $[0, \frac{1}{2}]$ that is a metric arc with Haantjes curvature 0 at every point, and that does not have Alt curvature at any point. [For definitions of the terms see Chapter III of the reviewer's "Theory and application of distance geometry", Oxford, 1953; these Rev. **14**, 1009]. *L. M. Blumenthal.*

Haupt, Otto. Zur Kennzeichnung der Kurven n-ter Ordnung im n-dimensionalen projektiven Raum. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1953**, 289–299 (1954).

A curve (an arc) is the topological image of a circle (an interval) in projective n-space. Its order is the maximum number of points in which it meets a linear $(n-1)$-space. The order of a point on a curve C is the order of a small

neighbourhood. If C is differentiable in a certain sense, its intersections with a linear subspace can be provided with multiplicities. Generalizing a theorem of Möbius, Hjelmslev proved: Suppose C is differentiable, every point of C has the order n, and C meets every $(n-2)$-space with a total multiplicity $\leqq n-1$; then C has the order n [Acta Math. 87, 59–82 (1952); these Rev. 13, 982]. The present paper announces an extension of Hjelmslev's theorem to non-differentiable C's and outlines a proof for the case $n=3$. The requirement that no straight line meets C more than twice excludes not only triples of collinear points on C and one-dimensional paratingents which meet C again but also a third class of straight lines whose definition is too involved to be quoted. *P. Scherk* (Saskatoon, Sask.).

Diamantopoulos, Th. Sur le rayon de contraction des courbes. Bull. Soc. Math. Grèce **28**, 81–100 (1954). (Greek. French summary)

For a plane curve $p(t)$ intersect the chord through $p(t+\Delta t)$ and $p(t-\Delta t)$ with the tangent at $p(t)$. The limit of this intersection for $\Delta t \to 0$ is called the center of contraction of $p(t)$ at t, and its distance from $p(t)$ (with a proper sign) is the radius of contraction. This concept depends, of course, on the parameter t. It is shown that the radius of contraction shares various properties with the radius of curvatures. As example for the applications we mention: If z_1 is a regular point of the complex analytic function $f(z)$ and $f'(z_1) \neq 0$, and if the centers of contraction of a one-parameter family of curves through z_1 lie on a conic with focus z_1, then the same holds at $w_1 = f(z_1)$ for the images of these curves under the mapping $w = f(z)$. *H. Busemann* (Göttingen).

Floras, Milt. Sur les intégrales premières de l'équation différentielle des lignes géodésiques d'une surface. Bull. Soc. Math. Grèce **28**, 107–114 (1954). (Greek. French summary)

The equation of the geodesics on a surface is considered in the form $\Delta\theta = 1$, where Δ is Beltrami's first differential operator. If in $\Delta\theta = 1$ we replace $\partial\theta/\partial u$ by p and $\partial\theta/\partial v$ by q, then a complete integral has the form $\varphi(u, v, p, q) = \alpha$, where p and q are related by $\Delta\theta = 1$ and $\partial p/\partial v = \partial q/\partial u$. The following fact (which is really already contained in Darboux's results) is proved: if $\lambda(u, v)$ is a solution of $\Delta\theta = 1$, which is contained in the complete integral, but is not itself complete, then $\varphi(u, v, \partial\lambda/\partial u, \partial\lambda/\partial v) = c$ is a family of geodesics with $\lambda(u, v) = c$ as orthogonal trajectories. Several applications are given, among which we mention: If $\lambda_i(u, v)$, $i = 1, 2, 3, 4$, are 4 solutions of $\Delta\theta = 1$ and if to the family $\lambda_i(u, v) = c$ of parallel curves there corresponds the family

$$\alpha(u, v)(\partial\lambda_i/\partial u)^2 + 2\beta(u, v)(\partial\lambda_i/\partial u)(\partial\lambda_i/\partial v) + \gamma(u, v)(\partial\lambda_i/\partial v)^2 = c$$

of geodesics, then $\alpha p^2 + 2\beta pq + \gamma q^2 = c$ is a first integral and the surface is a Liouville surface. *H. Busemann.*

Bagchi, Haridas. Note on systems of equi-potential n-surfaces (especially, n-ellipsoids) in an $(n+1)$-space. Bull. Calcutta Math. Soc. **46**, 25–28 (1954).

A focus of a surface in $(n+1)$-space is defined as a point whose isotropic cone has double contact with the surface. The complete focal surface is shown to consist, in general, of $n+1$ separate $(n-1)$-surfaces. The concept of confocal quadric is generalized from 3-space to $(n+1)$-space, and it is proved that a confocal set of $(n+1)$-ellipsoids is an isometric family. *S. B. Jackson* (College Park, Md.).

Şemin, F. Sur les sections d'une surface surosculées par leurs cercles de courbure. Rev. Fac. Sci. Univ. Istanbul (A) **19**, 34–44 (1954).

Take those plane sections of a surface S passing through a regular, non-umbilic point P which are superosculated by their circles of curvature, and consider the cone Σ enveloped by the planes in which these sections lie. In general this cone is of the fifth degree. But there are three directions in which the normal sections are superosculated by their circles of curvature, and if one of these directions should coincide with a principal direction or asymptotic direction, then the cone Σ will decompose into a cone of lower degree and one or more pencils of planes with axes in these special directions. A classification of the points on a surface at which these decompositions of Σ can occur is given, as is a classification of those surfaces for which the decomposition occurs at all regular, non-umbilic points. For instance, if the surface S is a cone of revolution, at each point except the vertex the cone Σ is composed of a cone of second degree and three pencils of planes, two of which have the rectilinear generator as a double axis, the third having the tangent to the parallel as axis.

Some of these results were obtained by Darboux [Bull. Sci. Math. (2) **4**, 348–384 (1880)] and A. Corio [Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Nat. **85**, 301–311 (1951); these Rev. **14**, 85]. The method used here is relatively simple, the starting point being the Laguerre function

$$L = \frac{d\rho}{ds}\cos\theta - \rho\sin\theta\left(\frac{3d\theta}{ds} + 2\tau\right),$$

where s, ρ, τ are the arc length, curvature and torsion of a curve C on S, and where θ is the angle between C's osculating plane and the normal to S. The function L has the same value for all curves with the same direction at P.

A. Schwartz (New York, N. Y.).

Lagrange, René. Sur les réseaux d'hélices de même cône directeur tracés sur certaines surfaces. Bull. Sci. Math. (2) **78**, 50–80 (1954).

Given a surface S, $x_\alpha = x_\alpha(u_i)$, $\alpha = 1, 2, 3$, $i = 1, 2$. Designate by Φ_1, Φ_2, Φ_3 its three fundamental forms, $\Phi_3 = 0$ describing the network of lines of curvature. For each fundamental form write the tensor

$$Q_{\alpha\beta} = H^{-2}[\phi_{22}\xi_{\alpha 1}\xi_{\beta 1} - \phi_{12}(\xi_{\alpha 1}\xi_{\beta 2} + \xi_{\alpha 2}\xi_{\beta 1}) + \phi_{11}\xi_{\alpha 2}\xi_{\beta 2}],$$

where H^2 is the discriminant of Φ_1 and $\xi_{\alpha i} = \partial x_\alpha/\partial u_i$. A network of helices on S which makes a constant angle ψ with a fixed unit vector $\mathbf{a}$ (abbreviated ψ, $\mathbf{a}$ in this review) will be conjugate with respect to one of the fundamental networks if $\sum Q_{\alpha\beta}\lambda_\alpha\lambda_\beta = 2\mu\cos^2\psi$, where the λ_α are the components of $\mathbf{a}$, $\mu = (2H^2)^{-1}(a_{11}\phi_{22} - 2a_{12}\phi_{12} + a_{22}\phi_{11})$, and the a_{ij} are the coefficients of the first fundamental form. When the problem of finding a surface which has such a network of helices cannot be solved geometrically, it can be reduced to the solution of a partial differential equation of the first order or of Monge-Ampère type.

A geometric argument shows that if S has an orthogonal network of helices ψ, $\mathbf{a}$, then S is the tangent developable of a helix. The skew helicoid is the only minimal surface which has a network of helices ψ, $\mathbf{a}$ conjugate with respect to Φ_2. Further, there are on the skew helicoid two such networks ψ_1, $\mathbf{a}$ and ψ_2, $\mathbf{a}$, and any network of helices ψ, $\mathbf{a}$ for which $\mathbf{a}$ is parallel to the axis of the helicoid will be conjugate with respect to Φ_2. To get the networks on the skew helicoid, take the intersections by cylinders of revolution through the axis

of the helicoid, each network being furnished by the cylinders of the same radius.

An arbitrary surface which has a network of helices ψ, $\mathbf{a}$ conjugate with respect to Φ_2 must be the affine transform of a minimal surface, and a representation for such surfaces is presented. As one interesting result, the author finds the surfaces, other than the skew helicoid, which can have three networks of helices ψ_r, $\mathbf{a}_r$, $r=1, 2, 3$, all conjugate with respect to Φ_2. The $\mathbf{a}_r$ are mutually orthogonal and the ψ_r satisfy $\sum \cos^2 \psi_r = 1$. Finally, the surfaces which have a network of helices ψ, $\mathbf{a}$ bisected by the lines of curvature (conjugate with respect to Φ_2) are the surfaces whose level lines are lines of curvature (les surfaces moulure).

The author does not claim that all of the results are new, but the point of view here presents both the new and the old results as an organized whole. *A. Schwartz.*

***Strubecker, Karl. Alcune applicazioni della geometria differenziale dello spazio isotropo.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 322–331. Edizioni Cremonese, Roma, 1954. 4000 Lire.

The author here considers the differential geometry of isotropic 3-space, that is, 3-space for which the element of arc is defined by $ds^2=dx^2+dy^2$. Special attention is paid to the correspondence of the surface elements in the isotropic space with pairs of points in the xy-plane. Relationship of this work to relative differential geometry is mentioned. Quadratic differential forms for surfaces in isotropic space are defined, and analogues of ordinary Euclidean results on curvature and minimal surfaces developed.

S. B. Jackson (College Park, Md.).

Pinl, Max. Integrallose Darstellung isotroper Kurven im sphärischen drei- und vierdimensionalen Raum. Math. Ann. **128**, 49–54 (1954).

Stereographic projection of Euclidean n-space into spherical n-space takes isotropic curves into isotropic curves. Since integral-free representations of isotropic curves in Euclidean 3-space and 4-space are known, this readily yields integral-free representations of isotropic curves in spherical 3-space and 4-space. *S. B. Jackson.*

Naumann, Herbert. Über das zweite Distributivgesetz im Zusammenhang mit den Viergeweben von Herrn R. Artzy. Math. Ann. **128**, 92–94 (1954).

In a recent paper by Artzy [Math. Ann. **126**, 336–342 (1953); these Rev. **15**, 346] plane 4-webs of the third kind are discussed, and it is shown that certain assumptions on the web are equivalent to various algebraic properties of a suitably introduced coordinate system. This is here extended by showing that the right-hand distributive law is equivalent to a special form of Desargues' theorem. In case all the assumptions on the web are satisfied, the coordinate algebra is a (skew) field. *S. B. Jackson* (College Park, Md.).

***Hopf, Heinz. Zur Differentialgeometrie geschlossener Flächen im Euklidischen Raum.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 45–54. Edizioni Cremonese, Roma, 1954. 4000 Lire.

A discussion of problems and results principally on surfaces with constant curvature and Weingartan surfaces of genus 0. The results are found in H. Hopf and K. Voss [Arch. Math. **3**, 187–192 (1952); these Rev. **14**, 583] and in H. Hopf [Math. Nachr. **4**, 232–249 (1951); these Rev. **12**, 634]. It is shown that in the latter paper the assumption of analyticity of the surface may be replaced by that of the analyticity of the relation between the principal curvatures.

H. Busemann (Göttingen).

***Sauer, R. Über Flächenklassen, bei denen sämtliche infinitesimale Verbiegungen durch Quadraturen darstellbar sind.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 122–129. Edizioni Cremonese, Roma, 1954. 4000 Lire.

The problem of determining infinitesimal bendings of a surface in 3-space can be reduced to the problem of solving a linear second-order partial differential equation, called the bending equation. This paper discusses two classes of surfaces for which all solutions of the bending equation are expressible in integral-free form and hence for which all bending surfaces are expressible in quadratures. These are surfaces for which the projections of the asymptotic lines form a displacement net (Rückungsnetz) either real or conjugate imaginary. *S. B. Jackson.*

Rešetnyak, Yu. G. Isothermal coordinates in manifolds of bounded curvature. Doklady Akad. Nauk SSSR (N.S.) **94**, 631–633 (1954). (Russian)

Let R be a surface of bounded curvature and M a subset of R homeomorphic to a circle whose boundary has a geodesic curvature of bounded variation. Then in M coordinates x, y can be introduced such that the line element of R takes the form $\lambda(x, y)(dx^2+dy^2)$. With $z=x+iy$ the function $\lambda(x, y)=\lambda(z)$ is given by

$$\lambda(z)=\exp\left\{\frac{1}{\pi}\int_M\int \log\left(|z-\zeta|^{-1}\right)\omega(dE_\zeta)+h(z)\right\},$$

where $\omega(dE_\zeta)$ is the integral curvature of the arc element dE_ζ of M at ζ and $h(z)$ is a harmonic function.

H. Busemann (Göttingen).

Jha, P., and Chariar, V. R. On a certain rectilinear congruence. Math. Student **22**, 77–83 (1954).

Les auteurs associent les différents points P d'une courbe donnée Γ aux différentes courbes $\gamma(P)$ d'un système ∞^1 tracées sur une surface donnée S, et étudient la congruence formée par l'ensemble des génératrices de cônes $[P, \gamma(P)]$. Ils déterminent les éléments fondamentaux (foyers, développables, paramètre de distribution) de la congruence envisagée, avec considération spéciale du cas où la congruence est normale, et de celui où, la congruence étant normale, ses rayons sont tangents à S. *P. Vincensini* (Marseille).

***Finikoff, G. Systèmes de congruences W.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 312–321. Edizioni Cremonese, Roma, 1954. 4000 Lire.

Dans cet article l'auteur analyse de façon approfondie la notion de systèmes de congruences W, introduite par lui dans un essai de généralisation de la notion de stratifiabilité ordinaire. Les systèmes de congruences W sont constitués par des ensembles de deux familles de ∞^1 surfaces liées par un famille de congruences W. Il en est ainsi des systèmes qui s'introduisent dans la théorie des transformations asymptotiques des surfaces de L. Bianchi, particulièrement étudiées par Finikoff, Terracini et Tortorici, et aux quelles Terracini a donné le nom de systèmes de Bianchi. Le premier exemple de système de congruences W, au sens étendu que l'auteur a donné à cette expression, a été donné par l'auteur en 1941 en généralisant la notion de stratifiabilité de Fubini

par la considération de deux multiplicités de ∞^1 plans. Sous l'impulsion de Finikoff, le problème de la recherche des systèmes de congruences W a fait l'objet de travaux importants de la part d'un certain nombre d'élèves du géomètre Russe, qui, par leurs efforts combinés, ont apporté au sujet une riche variété de résultats du plus haut intêret, travaux dont Finikoff donne un aperçu dans la note actuelle, qui reproduit une conférence faite par lui au colloque de géomètrie différentielle tenu à Pise en septembre 1953 pour honorer la mémoire de L. Bianchi. *P. Vincensini.*

*Klingenberg, Wilhelm. **Sui sistemi di sfere nella geometria di Laguerre.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 185–199. Edizioni Cremonese, Roma, 1954. 4000 Lire.

L'auteur commence par rappeler les principes de la représentation de l'espace L_3 à trois dimensions de Laguerre, dont les éléments sont les sphères orientées (L-sphères) de coordonnées x_i [$i=0, 1, 2, 3$; x_0=rayon, x_1, x_2, x_3=coordonnées cartésiennes ordinaires du centre] ou les plans orientés de l'espace ordinaire, sur l'espace M_4 à quatre dimensions à métrique indéfinie [$\delta^2=-(x_0-y_0)^2+\sum_{i=1}^{3}(x_i-y_i)^2$].

Il étudie ensuite les systèmes (variétés à deux paramètres) de L-sphères par celles des surfaces représentatives V_2 dans M_4. Dans cette étude les deux systèmes de droites (conjuguées par rapport à la sphère absolue de l'hyperplan infini H_3) intersections du plan tangent et du plan normal en un point quelconque avec l'hyperplan H_3, systèmes qui constituent les images tangentielle et normale de V_2, jouent un rôle important, en ce qu'ils permettent des traductions suggestives des propriétés de V_2 et par suite des systèmes de L-sphères correspondants, dans l'hyperplan infini H_3.

Les congruences de sphères de Ribaucour sont spécialement étudiées. Les surfaces qui leur correspondent dans M_4 ont fait l'objet de recherches de la part de W. Blaschke. L'auteur retrouve quelques uns des résultats de Blaschke, et en signale de nouveaux, relatifs notamment au cas où la courbure Riemannienne de la V_2 de Ribaucour envisagée est nulle. Et ce dernier cas le conduit à un rapprochement intéressant avec les surfaces minima de M_4.

P. Vincensini (Marseille).

Geldel'man, R. M. Stratification of k-parameter families of $(k-1)$-dimensional planes. Mat. Sbornik N.S. **34(76)**, 499–524 (1954). (Russian)

The space considered is a projective P_n, $n \geqq 2k-1$, so that for $k=2$, $n=3$, the classical case appears. In the first part of this paper we find the case of the one-sided stratification of a pair of ∞^2 families of straight lines in P_n; one-sided, since the non-trivial two-sided stratification is only realized for $n=2k-1$, and for $k=2$ the geometrical structure of stratified pairs is already determined by the one-sided stratification. Part II deals with the general case. If a pair of these families is here stratified (one-sidedly), then both families are special types of focal families, having $k(k-1)$-parametric focal subfamilies, which intersect all stratified surfaces in a k-conjugated system of lines. Special cases are discussed, as well as two-sided stratification in a P_{2k-1}. The investigation follows the now well-tested method of exterior forms which Finikov has introduced into this domain of geometry. *D. J. Struik* (Cambridge, Mass.).

Bell, P. O. A theorem on generalized conjugate nets in projective n-space. Duke Math. J. **21**, 323–327 (1954).

A theorem of the author [Proc. Amer. Math. Soc. **3**, 300–302 (1952); these Rev. **13**, 775] on a conjugate net in a linear space S_n of n ($\geqq 3$) dimensions is extended to the case of a generalized conjugate parametric net on an analytic variety of dimension r ($\leqq n-1$) in S_n defined by E. Cartan. *C. C. Hsiung* (Bethlehem, Pa.).

*Čech, E. **Deformazione proiettiva di strati d'ipersuperficie.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 266–273. Edizioni Cremonese, Roma, 1954. 4000 Lire.

An outline of the theory of projective deformations of one-parameter families of hypersurfaces is given with results only stated. *C. C. Hsiung* (Bethlehem, Pa.).

Hopf, Heinz. Sulla geometria riemanniana globale della superficie. Rend. Sem. Mat. Fis. Milano **23** (1952), 48–63 (1953).

The author discusses relations between the sign of the Gaussian curvature K and global properties of surfaces using various methods of proof of the following two classical theorems for closed surfaces of genus g as a basis of his development. (A) If $K>0$ everywhere then $g=0$. (B) If $K \leqq 0$ everywhere then $g \geqq 1$.

The classical proof by the Gauss-Bonnet formula is briefly treated, following which three other general methods of proof are examined. These are (a) the use of the properties of geodesics, (b) the use of a theorem of Cohn-Vossen on bounds for the curvatura integra of complete open surfaces and (c) introduction of isothermal parameters and use of the maximum principle. In the first two cases complete, open surfaces are included in the discussion and for such surfaces several interesting theorems result, for example (i) if a surface F may be divided into two parts F_1 and F_2 by a closed curve C such that neither $F_1 \cup C$ nor $F_2 \cup C$ is compact, then $K \leqq 0$ at some point; (ii) if $K>0$ everywhere on an open surface it is homeomorphic to the plane; (iii) there is no minimal closed curve on a surface for which $K>0$ everywhere, etc.

The paper concludes with a brief paragraph on the possibility of applying similar methods in higher dimensional cases. *W. M. Boothby* (Evanston, Ill.).

de Vries, Hans Ludwig. Über Riemannsche Räume, die infinitesimale konforme Transformationen gestatten. Math. Z. **60**, 328–347 (1954).

Consider a Riemannian space with a metric which is not necessarily positive definite but admits one or more infinitesimal conformal transformations on itself with or without further conditions upon the infinitesimal transformations. The purpose of this paper is to determine a canonical form of the line element for such a space. The conditions which are imposed on infinitesimal transformations are that the trajectories of the infinitesimal transformations constitute normal congruences or geodesic congruences. The most remarkable result is that the author can completely determine the canonical forms of line elements of Riemannian spaces which admit r independent infinitesimal conformal transformations whose trajectories all constitute geodesic congruences. *S. Sasaki* (Sendai).

Jankiewicz, C. Sur les espaces riemanniens dégénérés. Bull. Acad. Polon. Sci. Cl. III. **2**, 301–304 (1954).

This note is concerned with building up of an affine connection in a metric space V_n whose fundamental tensor $g_{\alpha\beta}$ ($\alpha, \beta=1, \cdots, n$) is of rank $m<n$. The number $d=n-m$ is called the deficiency of the space. The case of $d=1$ was

studied by Norden [Affinely connected spaces, Gostehizdat, Moscow-Leningrad, 1950, §76, 77; these Rev. 12, 441].

In this case $g_{\alpha\beta}g^{\beta}=0$ admits d linearly independent solutions $g_B{}^A$ $(A, E=m+1, \cdots, n)$ which may be chosen, in a special coordinate system, so that $|g_B{}^A|\neq 0$ while $|g_{ab}|\neq 0$ $(a, b=1, \cdots, m)$.

It is in this coordinate system that the author constructs an affine connection (asymmetric) so that $g_{\alpha\beta;\gamma}=0$ and $g^{\beta}{}_{B;\gamma}=0$.

Obviously this connection will depend on a number of arbitrary functions, $s^{\alpha}{}_{\alpha\beta}=\Gamma^{\alpha}{}_{[\alpha\beta]}$, $\Gamma^{A}{}_{ab}$ and $g_B{}^A$. The spaces of more interest are the reducible ones, those for which $|g_{\alpha\beta}|=\begin{vmatrix} g_{ab} & 0 \\ 0 & 0 \end{vmatrix}$. A necessary and sufficient condition for reducibility is $g^{\alpha}{}_{[A}g^{\beta}{}_{B];\alpha}+g^{\beta}{}_{[A;B]}=S^{E}{}_{AB}g^{\beta}{}_{E}$. In case $d=1$ the space is necessarily reducible. *M. S. Knebelman.*

Matsumoto, Makoto. Local imbedding of Riemann spaces. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **28**, 179–207 (1954).

The author considers various problems in the local imbedding of Riemann spaces which arise from the work of T. Y. Thomas [Acta. Math. **67**, 169–211 (1936)] and that of the reviewer [Amer. J. Math. **61**, 633–644 (1939); these Rev. **1**, 28]. A Riemann space of n-dimensions is said to be of "class p" if the lowest dimensional Euclidean space into which it can be locally imbedded is of dimension $n+p$. A polynomial in the components of the curvature tensor is obtained whose vanishing is a necessary condition that a Riemann space be of class p $(2p<n)$. A relationship between this polynomial and the coefficients of the second fundamental forms permits an intrinsic definition of the type r of a Riemann space for even r.

A study is made of the possibility of imbedding a Riemann n-space in an $(n+1)$-space of constant curvature, the curvature K of which is not preassigned. When such an imbedding is possible, the value of K is determined. Similar treatment is given to the possibility of imbedding Einstein n-spaces and conformally flat n-spaces into $(n+1)$-spaces of constant curvature. Finally the imbedding of a Riemann n-space into an $(n+p)$-space of constant curvature is treated, and this problem is reduced to that of an imbedding in a flat space. *C. B. Allendoerfer* (Seattle, Wash.).

***Walker, A. G. Riemann extensions of non-Riemannian spaces.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 64–70. Edizioni Cremonese, Roma, 1954. 4000 Lire.

Riemann extensions of non-Riemannian spaces first made their appearance in a paper by E. M. Patterson and A. G. Walker [Quart. J. Math., Oxford Ser. (2) **3**, 19–28 (1952); these Rev. **13**, 985]. In that paper the authors were concerned with a problem in the theory of parallel fields of null n-planes in a Riemannian $2n$-space, and the general concept of a Riemann extension was not their primary concern. In the paper under review the author places the theory of Riemann extensions upon an independent footing, relating it to the general problem of imbedding one manifold M in another, M', so that some structure on M' induces in a natural way a given structure on M. He concerns himself specially with the case when M is an n-dimensional manifold (of given structure) and M' a $2n$-dimensional Riemannian manifold R^{2n} which is the covariant tangent bundle of M. Thus M' is defined as a manifold, the essential problem being to assign to it a Riemannian metric in such a way that it determines the given structure of M. One method of doing this is suggested by the Patterson-Walker investigation on parallel fields of null planes (see the above reference), the relevant results of which are discussed and generalised in the paper now under review.

As an illustration of his general remarks, the author proceeds to examine in some detail the case when M is a manifold A^n of symmetric affine connection. A point z of R^{2n} is described by coordinates $(x^{\alpha}, \xi_{\alpha})$ $(\alpha=1, 2, \cdots, n)$ where x^{α} are local coordinates of a point x of A^n and ξ_{α} are the components of a covariant vector at x. The metric of the general Riemann extension R^{2n} of A_n is then given by

$$ds^2=(c_{\alpha\beta}-2\Gamma^{\gamma}{}_{\alpha\beta}\xi_{\gamma})dx^{\alpha}dx^{\beta}+2dx^{\alpha}d\xi_{\alpha},$$

where the $c_{\alpha\beta}$ are functions of the x^{α} and $\Gamma^{\gamma}{}_{\alpha\beta}$ are the coefficients of connection in A^n. In the fibre bundle with space R^{2n} the projection $p: R^{2n}\to A^n$ maps a vector or tensor at a point z of R^{2n} on a vector or tensor at the point $x=p(z)$ of A^n. A non-zero tensor at z may, however, have zero projection at x, and this fact leads the author to define "pseudo-projection," which can operate on non-zero tensors at z to give non-zero tensors at x. For example, the curvature tensor $R_{hij\bar{k}}$ at z has zero projection but has the curvature tensor $B^{\alpha}{}_{\beta\gamma\delta}$ of A^n as pseudo-projection. It follows that, if R^{2n} is flat, so is A^n. Other theorems illustrating the relation between A^n and R^{2n} are: (i) If the general extension of A^n is conformally flat, then A^n is projectively flat; (ii) if the Riemann extension of A^n is Cartan-symmetric, then A^n is symmetric; (iii) if A^n is simply harmonic, so is the general Riemann extension of A^n. Of these theorems, (ii) has been proved independently by Z. Afifi and the author, and (iii) is due to E. M. Patterson [J. London Math. Soc. **27**, 102–107 (1952); these Rev. **13**, 986]. *H. S. Ruse* (Leeds).

Eisenhart, Luther P. Generalized Riemann spaces and general relativity. II. Proc. Nat. Acad. Sci. U. S. A. **40**, 463–466 (1954).

This paper follows three former publications by the same author [same Proc. **37**, 311–315 (1951); **38**, 505–508 (1952); **39**, 546–551 (1953); these Rev. **13**, 279; **14**, 317, 1123] and is a continuation of the last one. Several identities are deduced and the possibility of solutions is discussed. There is a resulting identity of valence 4 in which the Einstein curvature tensor and a covariant derivative of the tensor of asymmetry appear. Contraction leads to a tensor of valence 2. *J. A. Schouten* (Epe).

***Pastori, Maria. Sullo spazio della recente teoria unitaria di Einstein.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 107–113. Edizioni Cremonese, Roma, 1954. 4000 Lire.

The asymmetric affine connection of the Einstein unified theory gives rise to two kinds of covariant differentiation, symbolized, in the case of a contravariant vector, by

$$A^{i}_{+;k}=\partial A^{i}/\partial x^{k}+A^{j}\Gamma^{i}{}_{jk},$$

$$A^{i}_{-;k}=\partial A^{i}/\partial x^{k}+A^{j}\Gamma^{i}{}_{kj}.$$

It has long been known that, given two infinitesimal vectors dx^i, δx^i at P, (x^i), then the same point is obtained (the fourth vertex of an elementary "parallelogram") whether δx^i is transported along dx^i by plus-parallelism or dx^i along δx^i by minus-parallelism. More recently, F. Graiff [Boll. Un. Mat. Ital. (3) **7**, 132–135 (1952); these Rev. **14**, 316] has suggested that the elementary 2-plane defined by dx^i, δx^i may be regarded as having two sides associated, respec-

tively, with the two sorts of parallelism. The present author finds this idea illuminating both geometrically and physically. Geometrically, it is to be observed that the vectors dx^i, δx^i define not only the 2-plane, but also the vector $S_{jk}^{\cdot\cdot i}dx^j\delta x^k$, where $S_{jk}^{\cdot\cdot i}$ is the torsion-tensor. This vector has opposite directions according as the one or the other parallelism is used, so may be made to correspond to the respective sides of the 2-plane. Physically, the author suggests, this may be regarded as corresponding to a space of dipoles.

The rest of the paper consists of a discussion of the rules of covariant differentiation in the Einstein theory, with descriptions of them in terms of the "two-sided" concept.

H. S. Ruse (Leeds).

Haantjes, J. On a special class of spaces A_n. Nieuw Arch. Wiskunde (3) **2**, 97–102 (1954).

Let A_n be an affinely connected space without torsion and consider a congruence of curves in A_n. Take an arbitrary path g in A_n and draw through every point on g curves which belong to the congruence in consideration. Then we get a two-dimensional surface C_2. The author studies first necessary and sufficient conditions in order that all C_2 be totally geodesic. They are given as follows: (1) If we represent the congruence of curves by a vector field v^i, then v^i is torse-forming (i.e. v^i satisfies a relation of the form $\nabla_j v^i=\lambda A_j^{\cdot i}+p_j v^i$); and (2) the curvature affinor satisfies a relation of the following form:

$$v^h R_{kji}^{\cdot\cdot\cdot h}=S_j v^h+r_j A_i^{\cdot h}+t_i A_j^{\cdot h},$$

where $A_i^{\cdot h}$ is the unit affinor. If C_2 is flat, then we can endow C_2 with a natural induced connexion without further assumption. The author proves secondly that the induced connexion is flat if and only if (1) and (3) $v^h R_{k(ji)}^{\cdot\cdot\cdot h}=0$ are satisfied. Recalling the theory of representation of spaces with projectively connected $(n-1)$-spaces in affinely connected n-spaces, the author regards curves of the given congruence in A_n as points of a new space H_{n-1}. The affine connexion in A_n defines a projective connexion in H_{n-1} if and only if the Lie derivative of the parameter of the affine connexion with respect to ρv^i is 0, where ρ is a suitable scalar. The author proves that if every C_2 is totally geodesic and flat with respect to the induced connexion, then the necessary and sufficient condition that the given affine connexion induces a projective connexion to H_{n-1} is that $p_j=\partial_j \log \lambda$.

The author finally applies his result to subprojective spaces getting the following theorem: An A_n $(n>3)$ is subprojective if and only if the vector v^i is torse-forming and the curvature affinor has the form

$$R_{kji}^{\cdot\cdot\cdot h}=2T_{[kj]}A_i^{\cdot h}-2A_{i[k}^{\cdot h}T_{j]i}+U_{kji}v^h.$$

This result contains as a special case the theorem of J. A. Schouten [Nederl. Akad. Wetensch. Proc. Ser. A. **56**, 72–79 (1953); these Rev. **14**, 1123] to the effect that "if the connexion of a nonprojectively euclidean A_n can be written in the form

$$\Gamma_{ji}^{\cdot\cdot h}\overset{\circ}{=}w_j A_i^{\cdot h}+w_i A_j^{\cdot h}+w_j v^h,\quad w_{ji}=w_{ij},$$

with respect to some coordinate system and if the vector field v^i is torse-forming, then the connexion is subprojective". The last theorem is an improved one of T. Adati [Tensor (N.S.) **1**, 105–115, 116–129, 130–136 (1951); these Rev. **14**, 317, 318], which corresponds to the same theorem of Adati [Tôhoku Math. J. (2) **3**, 159–173, 330–342, 343–358 (1951); these Rev. **14**, 86, 317] in the case of Riemannian geometry. The author also answers partially the question whether or not the condition "torse-forming" can be omitted from his last theorem stated above. This corresponds to the same question posed by J. A. Schouten about his theorem stated above.

S. Sasaki (Sendai).

Prasad, Ayodhya. Inter-relations of paths and affine connections in a non-Riemannian space. Bull. Calcutta Math. Soc. **46**, 29–36 (1954).

Four different arbitrary forms of paths are taken in a non-Riemannian manifold and in each case affine-connections are searched for. In the first case, it is found that the values of $\Gamma^{\alpha}_{\beta\gamma}$'s are zero. In the second case it is found that $\Gamma^{\alpha}_{\beta\gamma}$'s cannot be expressed in terms of the coordinates, which physically means that there exists no space with a symmetric connection such that equations of path can be as taken in this case. In the third case, $\Gamma^{\alpha}_{\beta\gamma}$'s are shown expressible in terms of coordinates, and their values under particular conditions are determined. Applying the necessary and sufficient condition for a space with an assigned symmetric connection to be Riemannian, a metric is evolved as a particular solution. In the last case dealing with the fourth form of the path, different sets of $\Gamma^{\alpha}_{\beta\gamma}$'s under different conditions, are obtained. The physical significance of this result is that the same form of the path is compatible with different affine-connections. (Author's summary.)

V. Hlavatý (Bloomington, Ind.).

Varga, O. Bedingungen für die Metrisierbarkeit von affinzusammenhängenden Linienelementmannigfaltigkeiten. Acta Math. Acad. Sci. Hungar. **5**, 7–16 (1954). (Russian summary)

The author formulates the differential system for an affine connection in the space of line elements to be: 1) metrisable; 2) the affine connection defined by Cartan for a Finsler metric. These differential systems are prolonged, and criteria are given for the existence of solutions satisfying the desired conditions.

S. Chern (Chicago, Ill.).

Egorov, I. P. Motions in affinely connected spaces. Doklady Akad. Nauk SSSR (N.S.) **87**, 693–696 (1952). (Russian)

Le travail contient la solution du problème suivant: On doit déterminer l'ordre du groupe des mouvements dans l'espace courbe à connexion affine sans torsion et qui n'est pas equiaffine. L'auteur prouve les theorèmes suivants. 1) L'ordre du groupe des mouvements dans l'espace considéré à n dimensions n'est pas plus que n^2-n+3. 2) Il n'existe aucun espace projectif-euclidien avec le tenseur de Ricci antisymétrique. 3) Le groupe des mouvements avec l'ordre maximum dans l'espace à connexion affine et non equiaffine est le groupe transitif, complet et son ordre est n^2-n+1. 4) L'espace à connexion affine et non equiaffine avec le groupe des mouvements d'ordre maximum est l'espace projectif-euclidien. 5) Il n'existe aucun espace à connexion affine avec le groupe complet et transitif des mouvements qui a l'ordre r pour lequel $n^2-n+1<r<n^2$. 6) L'ordre maximum du groupe intransitif des mouvements dans l'espace à connexion affine est égal à n^2-1. 7) Il n'existe aucun espace à connexion affine avec le groupe des mouvements complet, intransitif et qui a l'ordre r pour lequel on a $n^2-n+1<r<n^2-1$. Les démonstrations sont seulement indiquées.

F. Vyčichlo (Prague).

Egorov, I. P. On motions in spaces with an affine connection. Doklady Akad. Nauk SSSR (N.S.) **89**, 781–784 (1953). (Russian)

Dans le travail l'auteur montre la caractéristique géométrique 1) de l'espace A_n à connexion affine sans torsion et qui a la courbure différente de zéro et 2) de l'espace L_n à connexion (demisymétrique) avec le tenseur de torsion pour lequel on a $\Omega^{\alpha}{}_{\beta\gamma}=\delta_{\beta}{}^{\alpha}\Omega_{\gamma}-\delta_{\gamma}{}^{\alpha}\Omega_{\beta}$, où on a le groupe des mouvements d'ordre maximum. On y emploie soit des groupes complets et transitifs d'ordre n^2, soit des groupes complets, intransitifs d'ordre n^2-1. On connait les conditions nécessaires et suffisantes pourque l'espace A_n possède le groupe des mouvements d'ordre maximum [mêmes Doklady (N.S.) **84**, 433–435 (1952); ces Rev. **14**, 318]: 1) l'espace A_n est projectif-euclidien et equiaffine; 2) $R_{ij}=(1-n)\varepsilon\lambda_i\lambda_j$ ($\varepsilon=\pm 1$), $\lambda_{i,j}=a\lambda_i\lambda_j$, où le tenseur $\lambda_{i,j}$ est symétrique, λ_i est le gradient de la fonction λ et a est la fonction de λ dans le cas du groupe complet et intransitif, ou une constante dans le cas du groupe complet et transitif.

En partant des résultats ceux-ci l'auteur prouve les théorèmes suivants. 1) La condition nécessaire et suffisante pourque l'espace à n dimensions, à connexion affine sans torsion et avec la courbure différente de zéro soit l'espace A_n avec le groupe d'ordre maximum est: Ils existent dans A_n les $(n-1)$ champs indépendents des vecteurs contravariants absolument parallèles. 2) Dans l'espace A_n, avec le groupe d'ordre maximum, chaque hypersurface $\lambda(x^1,\cdots,x^n)=C$, où C est le paramètre, est totalement géodésique. 3) Les coefficients $\Gamma^{\alpha}{}_{\beta\gamma}$ de la connexion affine de l'espace A_n avec le groupe des mouvements d'ordre maximum, transitif et complet peuvent avoir la forme

$$\tfrac{1}{2}\Gamma^1{}_{11}=\Gamma^2{}_{21}=\cdots=\Gamma^n{}_{n1}=-\frac{x^1}{(x^1)^2+4\varepsilon},\quad \text{ou}\quad -\frac{x^1+1}{(x^1)^2+b}$$

(les autres $\Gamma^{\alpha}{}_{\beta\gamma}$ sont égaux au zéro) où $b\neq -1$ est la constante arbitraire. 4) Les coefficients $\Gamma^{\alpha}{}_{\beta\gamma}$ de la connexion affine de l'espace A_n avec le groupe d'ordre maximum, intransitif, complet, peuvent toujours avoir la forme:

$$\tfrac{1}{2}\Gamma^1{}_{11}=\Gamma^2{}_{21}=\cdots=\Gamma^n{}_{n1}=\alpha(x^1),$$

(les autres $\Gamma^{\alpha}{}_{\beta\gamma}$ sont égaux au zéro), où $\alpha(x^1)$ est une fonction de x^1. Dans le cas où l'espace est L_n, la connexion associée est

$$L^{\alpha}{}_{(\beta\gamma)}=-\delta_{\beta}{}^{\alpha}\psi_{\gamma}-\delta_{\gamma}{}^{\alpha}\psi_{\beta},$$

où

$$\partial_{\alpha\beta}\psi=-\partial_{\alpha}\psi\partial_{\beta}\psi+C_1\partial_{\alpha}\Omega\partial_{\beta}\Omega,$$
$$\partial_{\alpha\beta}\Omega=-\partial_{\alpha}\psi\partial_{\beta}\Omega-\partial_{\beta}\psi\partial_{\alpha}\Omega+C_2\partial_{\alpha}\Omega\partial_{\beta}\Omega,$$

et C_1, C_2 sont les constantes ou les fonctions de Ω ($\Omega_i=\partial_i\Omega$). On a les théorèmes: 5) Les vecteurs ψ_i, Ω_i de l'espace L_n avec le groupe d'ordre maximum, qui donnent la connexion euclidienne, peuvent avoir la forme

$$\psi_1=\psi_2=\cdots=\psi_n=0,\quad \Omega_1=1,\quad \Omega_2=\cdots=\Omega_n=0,$$

ou

$$\psi_1=\psi_2=\cdots=\psi_n=0,\quad \Omega_1=\frac{k}{x^1},\quad \Omega_2=\cdots=\Omega_n=0.$$

6) L'ordre maximum du groupe intransitif, complet, dans l'espace à connexion affine avec torsion, est n^2-1. Les vecteurs associés peuvent avoir la forme

$$\psi_1=\alpha(x^1),\quad \psi_2=\cdots=\psi_n=0,$$
$$\Omega_1=\beta(x^1),\quad \Omega_2=\cdots=\Omega_n=0,$$

où α, β sont les fonctions de x^1 arbitraires.

Le travail contient aussi les équations des groupes étudiés. *F. Vyčichlo* (Prague).

Horie, Nobuo. On the holonomy groups of the group-spaces. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **28**, 161–167 (1954).

Every Lie group is endowed with three connections in a natural way, the so-called plus-connection, the minus-connection, and the zero-connection. The latter has no torsion; and the former two have no curvature, but opposite torsion. The author proves that the restricted holonomy group of the "rollings" of the tangent space with respect to the plus-connection along null-homotopic paths consists of translations only, and that the linear subspace spanned by these translations is exactly the Lie algebra of the commutator subgroup. The theorem is a direct consequence of theorems 7 and 14 of a paper by the reviewer [see the two following reviews], but the proof for the special case dealt with in the paper under review is perhaps the simplest that could be given. *A. Nijenhuis* (Princeton, N. J.).

Nijenhuis, Albert. On the holonomy groups of linear connections. IA, IB. General properties of affine connections. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 233–240, 241–249 (1953).

Let $h(M,p)$ be the (homogeneous) holonomy group at the point p of an n-dimensional manifold M of class C^k ($k\geqq 3$) with an affine connection $\Gamma^{\alpha}{}_{\mu\lambda}$ and $h^0(M,p)$ the restricted (homogeneous) holonomy group, that is, the subgroup of $h(M,p)$ belonging to those curves which can be contracted to p. If p and q are any two points of M, then an isomorphism between $h(M,p)$ and $h(M,q)$ and also between $h^0(M,p)$ and $h^0(M,q)$ can be established by

$$a\to a'=\varphi a\varphi^{-1}\qquad \begin{matrix} a\in h(M,p) \text{ or } h^0(M,p)\\ a'\in h(M,q) \text{ or } h^0(M,q)\end{matrix}$$

where φ is a mapping of the tangent space at p into the tangent space at q defined by a curve connecting p and q. We call natural injection a mapping of any kind of holonomy group into any other holonomy group by parallel displacement along an appropriate curve.

If U is a connected open set of M and p a point of U, then we denote by $h^0(U,p)$ the restricted holonomy group of the manifold U, with base point p.

The author proves first the theorem: The restricted holonomy group $h^0(M,p)$ of M with base point p is a Lie subgroup of the centred affine group. Its Lie algebra $dh^0(M,p)$ is spanned by the matrices that arise from the $R_{\nu\mu}(x)$, $x\in M$, by parallel transport to p along any curves. Here $R_{\nu\mu}$ denotes the set of matrices determined by the curvature tensor $R^{\varkappa}{}_{\nu\mu\lambda}$ with $\varkappa$ and λ as column and row indices. He next defines the local holonomy group $h^*(p)$ at a point p by

$$h^*(p)=\bigcap_{U\ni p} h^0(U,p),$$

and proves: Every element of $h^0(M,p)$ is a finite product of elements of the groups $h^*(x)$, $x\in M$, after suitable natural injections; if $\dim h^*(x)$ is constant over M, then $h^0(M,p)=h^*(p)$; the local holonomy group $h^*(p)$ at p is a Lie subgroup of $h^0(M,p)$. Its Lie algebra $dh^*(p)$ is spanned by the matrices that arise from the $R_{\nu\mu}(x)$ in a suitable neighbourhood of p by parallel transport to p along a suitable set of curves (e.g., geodesics).

If M and the affine connection are of class C^{∞}, one can define at each point p a group called the infinitesimal holonomy group $h'(p)$. It is determined by the fact that its Lie algebra is spanned by the matrices

$$R_{\nu\mu}(p),\quad \nabla_{\omega}R_{\nu\mu}(p),\quad \cdots,\quad \nabla_{\omega_r,\cdots,\omega_1}R_{\nu\mu}(p),\quad \cdots,$$

and that it is a connected Lie subgroup of the centred affine group. The group $h'(p)$ is a subgroup of $h^*(p)$.

Concerning the relation between these three groups $h(M, p)$, $h^*(p)$ and $h'(p)$, the author proves: If $h'(x)=h^*(x)$ for all $x \in M$ of class C^∞, then $h^0(M, p)=h^*(p)=h'(p)$. If M and the connection are analytic, then $h^0(M, p)=h'(p)$.

The author studies also non-analytic cases. *K. Yano.*

Nijenhuis, Albert. On the holonomy groups of linear connections. II. Properties of general linear connections. Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indagationes Math. **16**, 17–25 (1954).

Let M be an n-dimensional manifold of class C^k ($k \geqq 1$) covered by a set Σ of neighbourhoods U with coordinate system (x), and associated with every point $x \in M$ let there be a linear N-dimensional real vector space F_x with a rectilinear coordinate system (A). We put $B=\bigcup_{x \in M} F_x$, π: $B \to M$ defined by $\pi(a)=x$ if $a \in F_x$. Then F_x, B and π are called respectively: fibre over x, bundle space and projection map. At every $x \in U \cap U'$ there is a matrix with elements $A_A^{A'}(x)=\partial x^{A'}/x^A$ and also a matrix $\|A_A^{A'}(x)\|$ belonging to the change of coordinates in F_x from (A) to (A'). If the family Σ has the property that for all U, $U' \in \Sigma$ the coefficients $A_A^{A'}(x)$ are functions of class C^l, $0 \leqq l \leqq k$, over $U \cap U'$, then the collection $\{B, M, \pi\}$ is called a linear fibre bundle of class $C^{k,l}$. The bundle $\{B, M, \pi\}$ is called linearly connected of class C^m, $m \leqq l-1$, if over every $U \in \Sigma$ there is defined a set of nN^2 real functions $\Gamma_{\mu B}^{A}(x)$, $x \in U$, of class C^m, in such a way that for every $x \in U \cap U'$, U, $U' \in \Sigma$, one has

$$\Gamma_{\mu' B'}^{A'}=A_{\mu'}^{\mu}(A_{AB}^{A'B}\Gamma_{\mu B}^{A}+A_{B'}^{B}\partial_{\mu}A_{B}^{A'}).$$

Using the functions Γ's, we can define a mapping of the F_p into the F_q along a curve joining p and q. This is called the parallel displacement along C. Thus, as in a previous paper (see the above review) we can define the holonomy group $\mathfrak{h}(M, p)$, the restricted holonomy group $\mathfrak{h}^0(M, p)$, local holonomy group $\mathfrak{h}^*(p)$ and the infinitesimal holonomy group $\mathfrak{h}'(p)$, provided $m=\infty$. The author proves the theorems analogous to those stated in the paper reviewed above.

He studies next the non-homogeneous holonomy groups $H(M, p)$ and $H^0(M, p)$ and gets the following interesting theorems. (1) Whenever $\xi^x(t)$, $0 \leqq t \leqq \epsilon$, is a curve of class D^1 in $E_n(p)$, $p \in M$, and $\xi^x(0)$ is the origin of $E_n(p)$, there exists an $\epsilon>0$ such that there is precisely one curve $x(t)$, $0 \leqq t \leqq \epsilon$, in M, whose image under development into $E_n(p)$ is the segment $\xi^x(t)$, $0 \leqq t \leqq \epsilon$. If M is a complete Riemann space of positive definite metric, then $\epsilon=1$. (2) If M is a manifold of symmetric affine connexion, and if $H'(p)$ is not trivial, then $H^0(M, p)$ has a nontrivial subgroup of affine rotations about the origin of $E_n(p)$. (3) If M has a symmetric affine connexion of C^∞, if $H'(x)$ is not trivial for any $x \in M$, and if M is complete, then for every point $\eta \in E_n(p)$ there is a nontrivial subgroup of $H^0(M, p)$ which leaves η invariant. (4) If M is a complete Riemann space of positive definite metric and if $H^0(M, p)$ leaves exactly one point invariant, then M is homeomorphic to a Euclidean n-space. (5) Whenever the holonomy group $H^0(M, p)$ of a space M with affine connexion contains a one-parameter subgroup of translations, $H^0(M, p)$ contains either all translations or $h^0(M, p)$ is reducible. *K. Yano* (Amsterdam).

Ambrose, W., and Singer, I. M. A theorem on holonomy. Trans. Amer. Math. Soc. **75**, 428–443 (1953).

Etant donnée une connexion infinitésimale sur un espace fibré principal différentiable E de groupe structural le groupe de Lie G, le groupe d'holonomie au point z de E est défini par l'ensemble des éléments g de G tels que z et zg puissent être joints par un chemin de E continûment différentiable par morceaux, horizontal relativement à la connexion. Si l'on se restreint aux chemins dont la projection sur la base est un lacet homotope à 0, on obtient le groupe d'holonomie restreint $\sigma(z)$. Soit L l'algèbre de Lie de G, $E(z)$ l'ensemble des points de E qui peuvent être joints à z par des chemins horizontaux continûment différentiables par morceaux; il est prouvé que $E(z)$ est une sous variété de E. Le théorème principal de ce travail est le suivant: si Ω est la forme de courbure de la connexion (à valeurs dans L), la sous-algèbre de L engendrée par les $\Omega(s, t)$ (s, t vecteurs tangents à E) aux différents points de $E(z)$ est l'algèbre de Lie de $\sigma(z)$. Ce théorème fondamental correspond dans le cas des connexions affines à un résultat établi en détail par A. Nijenhuis [dans les oeuvres reviewées ci-dessus]. Afin de l'établir les auteurs reprennent la théorie des connexions infinitésimales sur un espace fibré principal, développée par Ehresmann, Chern et Henri Cartan. Une démonstration intrinsique des "équations de structure" (au sens d'Elie Cartan) de la connexion est donnée au cours du papier au moyen d'un lemme nouveau. *A. Lichnerowicz* (Paris).

Igusa, Jun-ichi. On the structure of a certain class of Kaehler varieties. Amer. J. Math. **76**, 669–678 (1954).

A compact complex variety carrying a Kähler metric with constant curvature is called elliptic, hyperbolic or parabolic according as the curvature is positive, negative or zero. It is proved that the universal covering variety $\bar{V}^n$ of V^n is the complex projective space S^n when the space is elliptic, the interior of a sphere K^n when the space is hyperbolic, and the complex vector space C^n when the space is parabolic.

If V^n is elliptic, V^n itself coincides with S^n. If V^n is hyperbolic, it is a non-singular projective model, i.e. there is an analytic homeomorphism of V^n onto a non-singular variety in a complex projective space. If V^n is parabolic, it has a uniquely defined finite covering variety T^n which is a complex torus.

If V^n and V'^n are complex varieties such that one is a finite covering of the other, then either both are analytically homeomorphic to non-singular projective models, or else neither has this property. Any hyperbolic V^n is necessarily a minimal model (in the sense of A. Weil). A parabolic V^n is a minimal model if it is a non-singular projective model. Further properties of hyperbolic and parabolic varieties are discussed. *T. J. Willmore* (Liverpool).

Guggenheimer, Heinrich. Geometria pseudo-kähleriana. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **15**, 271–274 (1953).

[For part I see same Rend. (8) **14**, 220–222 (1953); these Rev. **15**, 63.] This note contains some formulae preliminary to a study of pseudo-kählerian manifolds on which there exists an operator C_k whose kth power is the operator denoted by C in the theory of Kähler manifolds.

W. V. D. Hodge (Cambridge, England).

***Guggenheimer, Heinrich. Topologia differenziale delle trasformazioni cremoniane e delle riemanniane di funzioni di più variabili complesse.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 222–228. Edizioni Cremonese, Roma, 1954. 4000 Lire.

An almost-complex manifold M is characterized by the existence of a differentiable transformation C, on differential forms $f^{(p)}$ of all orders p, satisfying $CCf^{(p)}=(-1)^p f^{(p)}$.

Using this, operators $\delta = CdC$ and $D = d\delta$ (and some more general ones not used in the sequel) are introduced. A necessary condition for M to be complex is $d\delta + \delta d = 0$; assuming this ("pseudo-complex manifold"), D satisfies $DD = 0$, and hence may be used to define cohomology groups of forms (isomorphic to the de Rham groups on a pseudo-Kaehler manifold). Let M and N be such manifolds, N regularly imbedded in M, $H^p(M)$ and $H^p(N)$ be the D-cohomology groups, and $H^p(M, N)$ be the D-cohomology of forms of M with vanishing tangential components at N; the following exact sequence arises:

$$\cdots \to H^{p-2}(N) \to H^p(M, N) \to H^p(M) \to H^p(N) \to \cdots.$$

In particular, if M is an algebraic surface arising from the application of a Cremona transformation to a singular variety, and N is the subset of M containing all points where the transformation is not one-to-one or where M has multiple points, this sequence implies:

$$H^2(M) \cong H^2(M, N) + H^2(N).$$

This formula is used to calculate the second Betti number of some particular such surfaces. Some problems are suggested which may be approached by this method. The first (Hopf) is the study of which topological manifolds admit a Kaehler metric with possible singularities; or further, the study of cases when a regular and a singular Kaehler metric are defined, but determine different cohomology classes. The method above extends to higher dimensions, but operators raising the degree by more than two are useful.

R. C. Gunning (Princeton, N. J.).

*Barthel, Woldemar. **Über Minkowskische und Finslersche Geometrie.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 71–76. Edizioni Cremonese, Roma, 1954. 4000 Lire.

The paper begins with a brief report on an earlier paper [Arch. Math. **4**, 346–354, 355–365 (1953); these Rev. **15**, 556]. In that paper it was shown that, in contrast to the Minkowskian case, the normal curvatures of a hypersurface at a point do not satisfy a theorem of Euler. Here it is indicated without proof that a theorem of Euler is valid for the curvatures of the "Hyperflächenstreifen", which are roughly defined as follows: two tangent hyperplanes at infinitely near points of a curve $x(t)$ on the hypersurface intersect in an $(n-2)$-dimensional linear space L_t. The metric of the surface defines in the tangent hyperplane at $x(t)$ a Minkowski metric, which serves as a local Minkowski metric at each point of L_t in the manifold formed by the various L_t. The Finsler manifold then defined is the Streifen of the hypersurface along $x(t)$, and Euler's Theorem holds for the normal curvatures of the different Streifen through a given point.

H. Busemann (Göttingen).

Barthel, Woldemar. Über die Minimalflächen in gefaserten Finslerräumen. Ann. Mat. Pura Appl. (4) **36**, 159–190 (1954).

A new formulation of the geometry of a Finsler space is presented which has certain advantages over earlier treatments. There is given a function $L(x, \dot{x})$ which is positive and positive homogeneous of the first degree in $\dot{x}$. The basic differentiable manifold is made into a fibre space by taking as the fibre the space of oriented line-elements $\dot{x}$ subject to the condition $x^{i'} = (\partial x^{i'}/\partial x^i)\dot{x}^i$. The following postulates are made. (1) The metric tensor is $g_{ij} = \frac{1}{2}\partial^2 L/\partial \dot{x}^i \partial \dot{x}^j$. (2) The form $g_{ij}(x, \dot{x})V^iV^j$ is positive definite. (3) The parallel displacement defined by:

$$DV^i = dV^i + V^h(\Gamma_h{}^i{}_k dx^k + C_h{}^i{}_k d\dot{x}^k)$$

leaves the length of every vector invariant. (4) $C_{hik} = C_h{}^j{}_k g_{ij}$ are symmetric in k and i. Hence $C_{hik} = \frac{1}{2}\partial g_{ki}/\partial \dot{x}^h$. (5) Let $l^i = \dot{x}^i/L(x, \dot{x})$ and write $T_{ak}l^k = T_{ak}$. Define $A_{ak} = LC_{ak}$, and $A_i = A^k{}_{ki}$. Define $\Gamma^*{}_k{}^i{}_h = \Gamma_k{}^i{}_h - A^i{}_{kj}\Gamma_0{}^j{}_h$. Define

$$V^i\Big|_k = \frac{\partial V^i}{\partial x^k} - L\frac{\partial V^i}{\partial \dot{x}^k}\Gamma_0{}^*{}_k{}^i{}_h + V^h\Gamma_k{}^*{}^i{}_h.$$

Then, by the postulate, $\Gamma^*{}_{ak} - \Gamma^*{}_{ki} = l_i A_{k/h} - l_h A_{k/i}$. In Cartan's formulation, by contrast, $\Gamma^*{}_{ak} - \Gamma^*{}_{ki} = 0$. (6) The $\Gamma_k{}^i{}_h$ are uniquely determined by $\Gamma_k{}^*{}^i{}_h$.

On this basis the theory of curves and hypersurfaces is developed. The autoparallel curves turn out to be extremals, and the autoparallel hypersurfaces to be minimal surfaces. There is a detailed consideration of the general variation of the surface integral of a hypersurface, including its first and second variations.

C. B. Allendoerfer (Seattle, Wash.).

Soós, Gy. Über Gruppen von Affinitäten und Bewegungen in Finslerschen Räumen. Acta Math. Acad. Sci. Hungar. **5**, 73–84 (1954). (Russian summary)

This paper makes a study of the group of affine transformations in an affinely connected space of line elements. Using Lie's derivative, conditions for an affine infinitesimal transformation are obtained. The author then proceeds to study the integrability conditions of the differential system so obtained. These results are applied to a Finsler space, and necessary and sufficient conditions are found for a group of affine transformations of a Finsler space to contain a subgroup of motions.

S. Chern (Chicago, Ill.).

Ingarden, R. S. Über die Einbettung eines Finslerschen Raumes in einem Minkowskischen Raum. Bull. Acad. Polon. Sci. Cl. III. **2**, 305–308 (1954).

The problem considered is the local, isometric imbedding of an n-dimensional Finsler space with $ds = F(x_\alpha, dx_\alpha)$ into an m-dimensional Minkowski space with $ds = M(dy_i)$. The imbedding is defined if functions $\varphi_i(x_\alpha)$ are determined such that $M((\partial\varphi_i/\partial x_\alpha)dx_\alpha) = F(x_\alpha, dx_\alpha)$. V. V. Vagner [Doklady Akad. Nauk SSSR (N.S.) **66**, 785–788 (1949); these Rev. **11**, 55] proved that the imbedding can be found when $m = 2n$. The present author obtains explicit formulas for such an imbedding.

C. B. Allendoerfer.

Kawaguchi, Akitsugu. A remark to the theory of areal spaces. Nieuw Arch. Wiskunde (3) **2**, 115–117 (1954).

This note concerns a theorem proved by Tandai [Tensor (N.S.) **3**, 40–45 (1953); these Rev. **15**, 254] to the effect that the only areal spaces which are of the metric class are spaces of Riemann, Finsler or Cartan. The proof given by Tandai depends upon a result due to Iwamoto which involves the third derivatives of the fundamental function F. The author proves the equivalence of the Iwamoto result to the vanishing of the ecmetric tensor introduced by him and Tandai [Tensor (N.S.) **2**, 47–58 (1952); these Rev. **14**, 586], and hence deduces that the third derivatives of the F are not essentially involved in the theorem proved by Tandai.

E. T. Davies (Southampton).

Pidek, H. Sur un problème de l'algèbre des objets géométriques de classe zéro dans l'espace X_1. Ann. Polon. Math. **1**, 114–126 (1954).

If $x_1, \cdots, x_n$ are components of n geometric objects of type K in X_1 the question arises whether a function

$f(x_1, \cdots, x_n)$ can be a component of an object of the same type. In a former publication [Ann. Soc. Polon. Math. 24, no. 2, 111–128 (1954); these Rev. 15, 899] the answer was given for $n=1$ and $n=2$. Here the general case is dealt with. The first theorem establishes a form for the function $\phi(x, \xi)$ occurring in the transformation formula if f is a component of class C^0 of a geometric object of class K^1. The second theorem gives the form of f if ϕ has the form from the first theorem. The third theorem deals with a special form of ϕ and in the fourth theorem it is proved that a function f of the form given in the second or third theorem is under certain restrictions always a component of a geometric object.

J. A. Schouten (Epe).

Pidek, H. Sur un problème de l'algèbre des objets géométriques de classe zéro dans l'espace X_m. Ann. Polon. Math. 1, 127–134 (1954).

The results of Ann. Soc. Polon. Math. 24, no. 2, 111–128 (1954) [these Rev. 15, 899] and the paper reviewed above are generalized for X_m. The three theorems are the generalizations of the four theorems of the preceding paper. The formulae remain simple. *J. A. Schouten* (Epe).

***Schouten, J. A. On the differential operators of first order in tensor calculus.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 1–7. Edizioni Cremonese, Roma, 1954. 4000 Lire.

In tensor calculus, the well-known differential operators of the first order, rotation, divergence and covariant differentiation are defined as follows:

rot.: $(p+1)\partial_{[\mu}v_{\lambda_1\cdots\lambda_p]}$; $v=p$-vector,

div.: $\partial_\mu \mathfrak{V}^{\mu\kappa_2\cdots\kappa_p}$; $\mathfrak{V}=p$-vector density of weight $+1$,

cov. diff.: $\nabla_\mu v^\kappa = \partial_\mu v^\kappa + \left\{ \begin{matrix} \kappa \\ \mu\lambda \end{matrix} \right\} v^\lambda$,

where $\left\{ \begin{matrix} \kappa \\ \mu\lambda \end{matrix} \right\}$ are Christoffel symbols. The author points out first an important difference between the first two and the third. To define rot. and div. we need only the fields operated on and nothing more; to define the covariant differentiation we need besides a field another field $g_{\lambda\kappa}$ from which $\left\{ \begin{matrix} \kappa \\ \mu\lambda \end{matrix} \right\}$ can be derived. The covariant derivative of v^κ can be considered as a differential concomitant of the two fields v^κ and $g_{\lambda\kappa}$. Consequently the covariant differentiation can be interpreted in two ways, first as an operator depending on $g_{\lambda\kappa}$ and working on v^κ and secondly as an operator depending on v^κ and working on $g_{\lambda\kappa}$. The Lie derivative which is a differential operator of the first order depending on a contravariant vector field v^κ and applicable to any geometric object can be also considered as a differential concomitant of the v^κ and the geometric object. The Lie derivative of a vector u^κ with respect to v^κ is given by

$$\mathfrak{L}_v u^\kappa = v^\mu \partial_\mu u^\kappa - u^\mu \partial_\mu v^\kappa.$$

The author gives a generalization of the Lie derivative which gives a differential concomitant of two arbitrary contravariant quantities. Recently A. Nijenhuis [Nederl. Akad. Wetensch. Proc. Ser. A. 54 = Indagationes Math. 13, 200–212 (1951); these Rev. 13, 281] found a new concomitant of two mixed quantities of valence two h_λ^κ and l_λ^κ:

$$O_{[\mu}h^\kappa_{\lambda]} = h_{[\mu}{}^\rho \partial_{|\rho|} l^\kappa_{\lambda]} - l_\rho^\kappa \partial_{[\mu} h^\rho_{\lambda]} + l_{[\mu}{}^\rho \partial_{|\rho|} h^\kappa_{\lambda]} - h_\rho^\kappa \partial_{[\mu} l^\rho_{\lambda]}.$$

The author says "The concomitant of Nijenhuis, though entirely new, is so simple that there certainly must be more concomitants", and he gives a hint to a possibility of finding some such examples and a well defined process to construct such concomitants. *K. Yano* (Amsterdam).

Norguet, François. Produit tensoriel et produit de composition des courants. C. R. Acad. Sci. Paris 239, 667–669 (1954).

The notions of the tensorial product and product of composition of distribution are extended to currents. Applications are made to the theory of functions of several complex variables, including a generalisation of the Cauchy integral. *W. V. D. Hodge* (Cambridge, England).

Varini, Bruno. Il calcolo tensoriale da un punto di vista elementare. Archimede 6, 45–52, 140–149 (1954).

Truesdell, C. Remarks on the paper, "The physical components of vectors and tensors." Z. Angew. Math. Mech. 34, 69–70 (1954).

This is an addendum to an earlier paper [same Z. 33, 345–356 (1953); these Rev. 15, 584]. Starting with a set of n contravariant vectors $e_\nu{}^i$ in n-space (i = vector index, ν = identifying label), the author defines $e_j{}^\rho$ by $e_\nu{}^i e_j{}^\nu = \delta_j^i$ (hence $e_i{}^\nu e_\rho{}^i = \delta_\rho^\nu$), and uses these two sets of vectors to define non-holonomic components of tensors, e.g., $a_\rho{}^\nu = e_i{}^\nu e_\rho{}^j a_j{}^i$. Given a connection $\Gamma^i{}_{jk}$ (not necessarily symmetric), he defines

$$\Gamma^\nu{}_{\rho\sigma} = e_i{}^\nu e_\rho{}^j e_\sigma{}^k \Gamma^i{}_{jk} - e_\sigma{}^k \partial_\rho e_k{}^\nu, \quad \partial_\rho = e_\rho{}^i \partial_i.$$

Then if $\mu^i{}_{,j} = \partial_j \mu^i + \Gamma^i{}_{jk}\mu^k$ indicates a covariant derivative, it follows that

$$\mu^\nu{}_{,\rho} = \partial_\rho \mu^\nu + \Gamma^\nu{}_{\rho\sigma}\mu^\sigma,$$

analogous results holding for tensors of all orders. It is noted that $\Gamma^i{}_{jk} = \Gamma^i{}_{kj}$ does not imply $\Gamma^\nu{}_{\rho\sigma} = \Gamma^\nu{}_{\sigma\rho}$. To get physical components in an orthogonal coordinate system with positive-definite fundamental tensor $g_{11}, g_{22}, \cdots$, he proposes to equate the matrix $e_\nu{}^i$ to the diagonal matrix with elements $(g_{11})^{-\frac{1}{2}}, (g_{22})^{-\frac{1}{2}}, \cdots$. As the author indicates, the method is a simplification, with a view to physical applications, of theory already known in more general form [cf. J. A. Schouten, Tensor analysis for physicists, Oxford, 1951, these Rev. 13, 493]. [The $\Gamma^\nu{}_{\rho\sigma}$ of the present paper appears more fundamental than the similar symmetrical quantity used by the reviewer for dynamical purposes; cf. J. L. Synge, Univ. of Toronto Studies, Appl. Math. Ser. no. 2 (1936).] *J. L. Synge* (Dublin).

NUMERICAL AND GRAPHICAL METHODS

***Table of sine and cosine integrals for arguments from 10 to 100.** National Bureau of Standards Applied Mathematics Series, No. 32. U. S. Government Printing Office, Washington, D. C., 1954. xv+187 pp. $2.25.

Reprint of a table published earlier [New York, 1942; these Rev. 4, 89].

***Abramov, A. A. Tablicy ln Γ[z] v kompleksnoĭ oblasti. [Tables of ln Γ[z] in the complex plane.]** Izdat. Akad. Nauk SSSR. Moscow, 1953. 333 pp. (1 insert). 30.20 rubles.

This unusual table gives the real and imaginary parts of $\log_e \Gamma(x+iy)$ for the rectangle $x=1(.01)2$; $y=0(.01)4$. These

80000 values are given to 6 decimals, cut down from 8 figure calculations. Second differences with respect to y are given also. These are mostly less than 50 in absolute value. A nomogram is provided to facilitate interpolation. Formulas are given for computing the function outside the basic rectangle of the table. The table was constructed by subtabulation starting with about 150 values obtained by Stirling's asymptotic series and using eight different interpolatory schemes of halving the mesh interval. Final values were differenced and rounded on a punched card tabulator. The table represents a very considerable amount of hand computing. The printing and quality of paper is good.

D. H. Lehmer (Berkeley, Calif.).

***Tables of Whittaker functions.** Numerical Computation Bureau, Tokyo, Rep. No. 8. The Tsuneta Yano Memorial Society, Tokyo, 1954. 39 pp. $1.00.

Table 1 gives 6D values of $M_{k,1/2}(x)$ and $M'_{k,1/2}(x)$ for $x=0(.1)5$ and $k=0(.05)1$ with second central differences (both in the x and k direction).

Table 2 gives 6D values of

$$u_k(x)=e^{x/2}x^{-k}W_{k,1/2}(x), \quad v_k(x)=e^{x/2}x^{-k}W'_{k,1/2}(x)$$

for $x=5(.2)25$, $x^{-1}=.01(.01).04$, $k=.05(.05).95$ with second central differences (both in the x and k directions). [In that part of the table where the variable is x^{-1}, read u_k and v_k for uk and vk, and, presumably, read 0.01, etc. for 0.1, etc.]

Table 3. Auxiliary table of Everett interpolation coefficients.

Table 4. Auxiliary table of $\log x$ and e^{-x}.

A. Erdélyi (Pasadena, Calif.).

Ishiguro, Eiichi, Yuasa, Sayoko, Sakamoto, Michiko, and Kimura, Tōsaku. Tables useful for the calculation of the molecular integrals. V. Nat. Sci. Rep. Ochanomizu Univ. 4, 176–191 (1954).

[For parts I–IV see these Rev. 14, 799; 15, 255.] Table XVII gives values of $\int_x^\infty e^{-t}t^{-1}dt$ to 10 or more significant figures for $.5\leqq x\leqq 71$ at intervals varying between .5 and 2. Table XVIII gives values of $\int_{-\infty}^x e^t t^{-1}dt$ (presumably the Cauchy principal value) for $2\leqq x\leqq 35.5$. Table XIX gives 5½ pages of values of $\int_1^\infty e^{-\alpha t}t^n dt$ [the definition in the paper contains a misprint] for $5.25\leqq\alpha\leqq 58.5$, and ranges of n varying from $0\leqq n\leqq 3$ to $0\leqq n\leqq 35$. Table XX gives 2 pages of $\int_1^\infty O_\tau(\lambda)e^{-\alpha\lambda}\lambda^m d\lambda$ for $5.25\leqq\alpha\leqq 35.5$ and various m: unfortunately, the value of τ ($=0$?) is not stated. Table XXI gives 6 pages of

$$\int_{-1}^{1} e^{-\alpha\mu}P_\tau^\nu(\mu)\mu^m(1-\mu^2)^{\nu/2}d\mu$$

[the definition in the paper contains a misprint] for assorted values of the parameters.

Since the functions in the last three tables have been computed from their recurrence formulas, the authors believe that "the last two or three figures" are often meaningless.

A. Erdélyi (Pasadena, Calif.).

Uhler, Horace S. Hamartiexéresis as applied to tables involving logarithms. Proc. Nat. Acad. Sci. U. S. A. 40, 728–731 (1954).

The first word in the title means the discovery and elimination of all errors. The device, used previously by the author, of checking tables of logarithms of integers by adding the entries and comparing the result with $\log n!$ is extended to the case of a table of $\log n!$. Here the sum gives the logarithm of the product

$$P(m)=\prod_{n=1}^{m}(n!)=\prod p_i^{e}$$

where p_i is the ith prime and $e=e(p_i, m)$. The determination of $e(p_i, m)$ is discussed in the case $m=1000$ and a complete table of $e(p_i, 1000)$ is given. The author might have noted also that

$$e(p, m)=(m+1)\sum_{\alpha=1}^{\infty}\left[\frac{m}{p^\alpha}\right]-\frac{1}{2}\sum_{\alpha=1}^{\infty}\left\{\left[\frac{M}{p^\alpha}\right]^2+\left[\frac{M}{p^\alpha}\right]\right\}p^\alpha$$

where $[x]$ denotes the greatest integer $\leqq x$.

These results are applied to a checking of the first thousand entries of Duarte's Nouvelles tables de $\log n!$ à 33 décimales depuis $n=1$ jusqu'à $n=3000$ [Kundig, Genève, 1927], with the discovery of a misprint in log 99! and a transposition in log 266!.

D. H. Lehmer.

Ledley, Robert S. A digitalization, systematization and formulation of the theory and methods of the propositional calculus. National Bureau of Standards, Washington, D. C., Rep. 3363, v+83 pp. (1954).

Ledley, Robert S. Mathematical foundations and computational methods for a digital logic machine. J. Operations Res. Soc. Amer. 2, 249–274 (1954).

These two papers are concerned with free Boolean algebras with n generators, from the standpoint of their applications, particularly in connection with digital computing machines. The first paper is the original report and gives the theory in detail; the second is a summary with some additional discussion and examples. The basic idea is that the elements of the algebra can be represented as sequences of binary digits; that sums, products, and complements of such vectors, and also the unit element, can be defined in an obvious fashion; and that the whole theory of finite Boolean algebras can be erected on such a basis. The theoretical part of this program is rather sloppily done: the theorems do not follow from the stated postulates (which characterize finite Boolean algebra in general, the specialization to free algebras being made on the basis of the informal discussion); the rule of modus ponens is confused with the statement $\vdash X(X\rightarrow Y)\rightarrow Y$ ["Ass" of Whitehead and Russell, Principia mathematica, v. 1, 2nd ed., Cambridge, 1925]; and other inelegancies of similar character occur with sufficient frequency to give the reviewer the impression of carelessness. Nevertheless the author does make a contribution to the technique. His conception, although apparently differing only trivially from other well known ones, gives enough novelty in point of view to suggest procedures which, in some cases at least, have practical advantages. This applies to the solution of equations and to the expression of an element as a simplest sum of products. Of special interest is a study of basis-transformations (i.e. changes from one set of generators to another). In connection with these he finds the "absolute simplest form" in which an element can be expressed; this does not mean the simplest form of expression in terms of given generators, but the simplest form which is possible when one is free to choose the generators suitably. At the ends of the two papers he considers some examples; these examples, which are different in the two papers, are of character similar to these given by E. Schröder [Vorlesungen über die Algebra der Logik, Bd 1, Teubner, Leipzig, 1890, Vorlesung 13]. At the close of the second paper there is a discussion of implementation.

H. B. Curry (State College, Pa.).

Vaughan, Hubert. Symmetry in central polynomial interpolation. J. Inst. Actuar. **80**, 63–68 (1954).

The scope of this article is indicated by the following excerpts: "We use the term 'central' to indicate that the formulae considered are understood to be applied so that the centre of each segment interpolated is the same as the centre of the data from which it is calculated. . . . We propose to show that certain properties, usually attributed to particular formulae, are in fact general, and can be deduced directly from the principle of symmetry."

T. N. E. Greville (Washington, D. C.).

Gonçalves, J. Vicente. Sur la méthode de Newton. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) **3**, 191–196 (1954).

This paper contains a discussion of the conditions on $f(x)$ necessary in the use of Newton's formula for the computation of the zeros of the function $f(x)$. A variation of Newton's formula is indicated. *E. Frank* (Chicago, Ill.).

Rutishauser, Heinz. Der Quotienten-Differenzen-Algorithmus. Z. Angew. Math. Physik **5**, 233–251 (1954).

The author's unifying quotient-difference (QD) algorithm is described in detail and significant properties are proved. The algorithm completes Aitken's extension of Bernoulli's method for getting zeros of polynomials [A. C. Aitken, Proc. Roy. Soc. Edinburgh **46**, 289–305 (1926)]. There are some numerical examples, and a full report on applications is promised later. Let

$$f(z)=\sum_0^\infty s_1^{(\nu)}/z^{\nu+1};\quad f_\nu(z)=\sum_0^\infty s_1^{(\nu+\nu)}/z^{\nu+1};\quad e_0^{(\nu)}\equiv 0.$$

In the QD-algorithm one computes (for $\nu=0, 1, \cdots$; $\sigma=1, 2, 3, \cdots$) an array comprising the quotients

$$q_\sigma^{(\nu)}=s_\sigma^{(\nu+1)}/s_\sigma^{(\nu)};$$

the differences $d_\sigma^{(\nu)}=q_\sigma^{(\nu+1)}-q_\sigma^{(\nu)}$; the modified differences $e_\sigma^{(\nu)}=d_\sigma^{(\nu)}+e_{\sigma-1}^{(\nu+1)}$; and the new s-series $s_{\sigma+1}^{(\nu)}=s_\sigma^{(\nu+1)}e_0^{(\nu)}$. The main theorems state: (I) $e_n^{(\nu)}=0$ (for all ν) if and only if $f(z)$ is rational with denominator of degree n; (II) if $f(z)$ is rational with poles of distinct absolute value, $|\lambda_1|>|\lambda_2|>\cdots$, then $\lim_\nu q_\sigma^{(\nu)}=\lambda_\sigma$.

The motivating application is to compute the eigenvalues λ_i of a finite matrix A directly from the Schwarz constants $s_1^{(\nu)}=x_0^*A^\nu y_0$ relative to given column vectors x_0, y_0, while avoiding any biorthogonalization process or any calculation of the characteristic polynomial of A. The constants of the QD-algorithm are expressed in terms of those of three other related algorithms: (1) Lanczos's [J. Research Nat. Bur. Standards **45**, 255–282 (1950); Proc. Second Symposium on Large-Scale Digital Calculating Machinery, 1949, Harvard Univ. Press, 1951, pp. 164–206; these Rev. **13**, 163, 589] for computing the characteristic polynomial from the Schwarz constants; (2) Lanczos's [loc. cit.] for biorthogonalizing the sequences $\{A^{*\nu}x_0\}$, $\{A^\nu y_0\}$; (3) the Hestenes-Stiefel conjugate-gradient algorithm [J. Research Nat. Bur. Standards **49**, 409–436 (1953); these Rev. **15**, 651]. The intermediate polynomials of Lanczos are shown to be approximating denominators of a continued-fraction expansion of $f_\nu(z)$. Thus the QD-algorithm transforms a power series into a continued fraction. *G. E. Forsythe.*

Rutishauser, Heinz. Ein infinitesimales Analogon zum Quotienten-Differenzen-Algorithmus. Arch. Math. **5**, 132–137 (1954).

The author [see the preceding review] developed an algorithm for the determination of eigenvalues from Schwarz constants. By this algorithm the J- or S-continued fraction corresponding to $\sum_0^\infty s_\nu/z^{\nu+1}$ can also be determined. This algorithm consists of a quotient-difference (Q-D) scheme analogous to the ordinary difference scheme of a series. Since an analogy exists and the derivatives of a function can be obtained from the ordinary difference scheme by a suitable limit process, this led to the study of the corresponding limit process for the Q-D algorithm. In this paper the limit process is considered, and also some applications of it. *E. Frank* (Chicago, Ill.).

Kappus, R. L'algorithme de Gauss modernisé et son application à des systèmes d'équations linéaires dégénérés ou mal ordonnés. O.N.E.R.A. Note Tech. no. **11**, i+33 pp. (1953).

The "modernized Gaussian algorithm" is the Cholesky-Banachiewicz condensed form of Gaussian elimination for solving linear systems expounded, for example, by R. Zurmühl [Matrizen, Springer, Berlin-Göttingen-Heidelberg, 1950, §23; these Rev. **12**, 73]. The publication reviewed is an exposition of the following: a particular form of the modernized Gaussian algorithm; the square root variation; calculation of the determinant; triangularization in matrix blocks; back-substitution and iterated elimination to increase the accuracy; searching for pivots; singular systems; with a "diagonal band". Matrix notation is used. There are 12 references and 7 illustrative tableaux.

G. E. Forsythe (Los Angeles, Calif.).

Lavrent'ev, M. M. On improvement of the accuracy of the solution of a system of linear equations. Doklady Akad. Nauk SSSR (N.S.) **92**, 885–886 (1953). (Russian)

Let the numbers x_j' solve the system $\sum_{i=1}^n a_{ij}x_i=b_j$ $(j=1, \cdots, n)$ with determinant Δ; and let x_j'' solve a varied system $\sum_{i=1}^n a_{ij}x_i=b_j+\epsilon_j$, where $|\epsilon_j|\leq\epsilon$. Assume $\sum_{i=1}^n a^2{}_{ij}=1$ for all j. Two results are announced: I.

$$|x_i''-x_i'|\leq e^{1/2}|\Delta|^{-1}n^{1/2}\epsilon$$

for all i. The estimate is asymptotically precise for large n and Δ^{-1}. II. If $|x'_{i+1}-x_i'|<\lambda n^{-1}$ $(i=1, \cdots, n-1)$, then $|x_i''-x_i'|\leq(6\lambda\epsilon)^{1/3}|\Delta|^{-2/3}\epsilon^{2/3}$ for all i. Result II improves result I when the solution x_j' is sufficiently smooth.

G. E. Forsythe (Los Angeles, Calif.).

Lavrent'ev, M. M. On an estimate of the accuracy of a solution of a system of linear equations. Doklady Akad. Nauk SSSR (N.S.) **95**, 447–448 (1954). (Russian)

The error of the solution of a linear system, bounded in the maximum-modulus metric in an earlier article [see the preceding review], is now bounded in the euclidean metric. Assume $\sum_{j=1}^n a_{ij}x_j=\epsilon_i$, with $\Delta=|\det(a_{ij})|$. Assume either (*) $\sum_{i=1}^n a^2{}_{ij}=1$, or (**) $\sum_{j=1}^n a^2{}_{ij}=1$. It is proved that $(\sum_i x_i^2)^{\frac{1}{2}}\leq\Delta^{-1}e^{\frac{1}{2}}(\sum_i \epsilon_i^2)^{\frac{1}{2}}$. With hypothesis (**) the estimate cannot be essentially improved, but with (*) the factor $e^{\frac{1}{2}}$ can be dropped. The known representation $A=HU$ (H self-adjoint and U unitary) is used to reduce the proof to the case of self-adjoint A. The estimate then follows after the author has proved that $\lambda>\Delta\epsilon^{-1}$, where λ is the least eigenvalue of A. *G. E. Forsythe* (Los Angeles, Calif.).

Lavrent'ev, M. M. On the accuracy of solution of systems of linear equations. Mat. Sbornik N.S. **34(76)**, 259–268 (1954). (Russian)

In §1 the author proves the result stated in the paper reviewed above. In §2 he bounds the change $\Delta r''$ in the solution of a linear system $\sum_{j=1}^n a_{ij}x_j=b_i$ due to replacement

of the a_{ij} by $a_{ij}+\epsilon_{ij}$, where $\sum_{i,j=1}^{n}\epsilon^2_{ij}<(\epsilon'')^2$. He proves that in the euclidean metric

$$|\Delta r''|<e(\sum b_i^2)^{1/2}\epsilon''\Delta^{-2}(1-e^{1/2}\Delta^{-1}\epsilon'')^{-1},$$

where $\Delta=|\det(a_{ij})|$.

What the author is doing in §3 is not clearly stated. He apparently proves some things which lead to result II of the second preceding review. This concerns the improvement of the precision of the solution of a system $\sum_j a_{ij}x_j=b_i$ for which the a_{ij} or b_i are subject to error, when it is known that the true solution should be smooth.

Reviewer's note: In the fourth displayed formula on p. 260 the factor $\Delta/e^{1/2}$ should read $e^{1/2}/\Delta$. *G. E. Forsythe.*

Gavrilov, Yu. M. On the convergence of iterative processes and criteria of sign-definiteness of quadratic forms. Izvestiya Akad. Nauk SSSR. Ser. Mat. **18**, 87–94 (1954). (Russian)

"Ordinary" and "Seidel" iterations are considered for solving a finite system of linear algebraic equations (1) $A_0X=k$, where $A_0=(\delta_{ij})$ is real symmetric with positive diagonal elements. Let $A_0=A_1+A_2$, where A_1 is the diagonal part of A_0. Two forms equivalent to (1) and suitable for iteration are considered: (2) $X=AX+f$, where $A=-A_1^{-1}A_2$, $f=A_1^{-1}k$; (3) $X=(E-A_{01})X+k_1$, where E=identity, $A_{01}=A_0/\max\delta_{jj}$, $k_1=k/\max\delta_{jj}$.

Three results are proved: I. The "ordinary" iteration based on (2) converges (i.e., all eigenvalues of A are less than 1 in modulus) if and only if both A_0 and A_1-A_2 are positive definite. II. The "ordinary" iteration based on (3) converges (i.e., all eigenvalues of $E-A_{01}$ are less than 1 in modulus) if and only if both A_0 and $2E-A_{01}$ are positive definite. III. Write $A_{01}=Q+P+Q'$, where Q, P are the subdiagonal, diagonal parts of A_{01}. Then the "Seidel" iteration based on (3) converges (i.e., all eigenvalues of $(E+P)^{-1}(E-P-Q')$ are less than 1 in modulus) if and only if A_0 is positive definite. Result III for (2) is due to Reich [Ann. Math. Statistics **20**, 448–451 (1949); these Rev. **11**, 136].

The convergence of these iterations is suggested as a test for definiteness of the quadratic forms for A_0, A_{01}, etc.

G. E. Forsythe (Los Angeles, Calif.).

Gavrilov, Yu. M. On the convergence of simple iterations and criteria of sign-definiteness of quadratic forms. Dopovidi Akad. Nauk Ukrain. RSR **1953**, 389–393 (1953). (Ukrainian. Russian summary)

This article is a Ukrainian translation with minor variations of the part of the author's article reviewed above in which result I is established and applied as a test for definiteness of the quadratic forms A_0 and A_1-A_2.

G. E. Forsythe (Los Angeles, Calif.).

***Givens, Wallace. Numerical computation of the characteristic values of a real symmetric matrix.** Oak Ridge National Laboratory, Oak Ridge, Tenn. Rep. ORNL 1574. vi+107 pp. (1954).

This publication discusses in detail the adaptation to automatic digital computers of two algorithms for solving the problem of the title previously outlined by the author in 1951 [Nat. Bur. Standards Appl. Math. Ser., No. **29**, 117–122 (1953); these Rev. **15**, 472]. The first algorithm converts a real symmetric matrix A of order n to an orthogonally congruent Jacobi (i.e., triple diagonal) matrix S defined by $s_{ij}=a_i$ $(j=i)$, $s_{ij}=b_i$ $(|i-j|=1)$, $s_{ij}=0$ $(|i-j|>1)$. The congruence is formed as the product of $(n-1)(n-2)/2$ rotations of coordinate 2-planes. The second algorithm, which is not uniquely determined by the first, recursively forms a Sturm sequence of polynomials from which the desired eigenvalues of S can be localized. The algorithms are compared with that of von Neumann, Goldstine, and Murray [Taussky and Todd, Math. Mag. **26**, 71–88 (1952); these Rev. **14**, 715] which continues the rotations until A is approximately diagonal. The algorithms are presented here in all detail, with proofs. It is shown incidentally that an arbitrary rotation matrix is the product of plane rotations in the coordinate planes. Other results cited or proved precisely bound the change in the set of n eigenvalues of A due to variations in A; bounds are given in both the euclidean and maximum modulus metrics.

There follows an exhaustive and tedious study of different methods of carrying out the algorithms with "digital" numbers, together with bounds for the resulting errors $\Delta\lambda_i$ in the computed eigenvalues. The analysis adheres to the spirit and conventions of von Neumann and Goldstine [Bull. Amer. Math. Soc. **53**, 1021–1099 (1947); these Rev. **9**, 471]. The results are applicable to most existing automatic digital computers, and are phrased in terms of round-off errors for the pseudo operations $\times$, $\div$, $\sqrt{}$, and R (=right shift). Then the estimates are specialized to the ORACLE computer (of precision 2^{-39}), for which it is found that for $n=100$ one can make all

$$|\Delta\lambda_i|<131{,}600\cdot 2^{-39}=1.004\cdot 2^{-32}$$

and

$$\left(\sum_{i=1}^{100}|\Delta\lambda_i|^2\right)^{1/2}<131{,}705\cdot 2^{-39}.$$

Specific flow charts, codes, and time estimates are given for the ORACLE. A separate errata sheet dated 20 May 1954 corrects some errors and states that the codes worked for $n=89$ with all $|\Delta\lambda_i|<10^{-9}$.

In addition to giving estimates for the specific algorithms discussed, the author examines the general concept of error in a digital computation, and shows how delicately the ultimate purpose of the computation affects the criteria of best digital approximation to a mathematical routine. This is well illustrated in choosing digital approximations to the sine and cosine defining a rotation. *G. E. Forsythe.*

Derwidué, L. La méthode de L. Couffignal pour la résolution numérique des systèmes d'équations linéaires. Mathesis **63**, 9–12 (1954).

This is an exposition in direct elementary terms of Couffignal's method for solving linear algebraic systems [L. Couffignal, Revue Sci. (Rev. Rose Illus.) **82**, 67–78 (1944); these Rev. **8**, 128]. The idea is to use Chiò's method to condense an augmented matrix of coefficients.

G. E. Forsythe (Los Angeles, Calif.).

Harman, Harry H. The square root method and multiple group methods of factor analysis. Psychometrika **19**, 39–55 (1954).

Cholesky's square root method of solving a system of linear algebraic equations or of inverting a matrix is described [see, e.g., P. Dwyer, Linear computations, Wiley, New York, 1951, p. 116; these Rev. **13**, 283]. There is a historical survey, with 18 references, of the "multiple group methods" of factor analysis. The common basic technique underlying these methods and the use of the square root method in them are described in connection with a 9-variable example. A table compares the notation of the present paper

with that of Holzinger [Psychometrika 9, 257–261 (1944); these Rev. 6, 162], Thurstone [Multiple-factor analysis, Univ. of Chicago Press, 1947; these Rev. 9, 47], and Guttman [Psychometrika 17, 209–222 (1952)].

G. E. Forsythe (Los Angeles, Calif.).

Burkhardt, Felix. Über spezielle lineare Gleichungssysteme mit der Eigenschaft $\lim_{r\to\infty}(\mathfrak{E}-\mathfrak{A})^r=$Nullmatrix. (Ein Beitrag zur Gleichungslehre der Praktischen Analysis.) Wissensch. Z. Univ. Leipzig. Math.-Nat. Reihe 1952/1953, 187–192 (1953).

Motivated by economic problems, the author considers finite linear systems $\mathfrak{A}\mathfrak{x}=\mathfrak{b}$ with the property of the title, where $\mathfrak{E}$=unit matrix. He presents assorted known material: (1) that $\mathfrak{x}=\mathfrak{b}+\mathfrak{M}\mathfrak{b}+\mathfrak{M}^2\mathfrak{b}+\cdots$, where $\mathfrak{M}=\mathfrak{E}-\mathfrak{A}$; (2) use of punch-card equipment including multipliers to evaluate the above series; (3) various sufficient conditions for $\mathfrak{M}^r\to 0$; (4) a few methods to solve linear systems numerically; (5) use of passive electrical networks or mechanical machines to solve linear systems. There are over a dozen references to the literature, mostly German.

G. E. Forsythe (Los Angeles, Calif.).

Wilkinson, J. H. The calculation of the latent roots and vectors of matrices on the pilot model of the A.C.E. Proc. Cambridge Philos. Soc. 50, 536–566 (1954).

Since the A.C.E. has no auxiliary storage, and only 361 words of fast storage, there was virtually no alternative to the use of iterative methods for finding latent roots. This paper summarizes very considerable experience at the National Physical Laboratory in the use of such methods for solving the latent-root and vector problem, with unsymmetric as well as with symmetric matrices. Many will be interested in the statement, made and emphasized, that even for matrices of order 50 or more it is possible to get all the roots without serious loss in accuracy, and, indeed, with greater accuracy than was attained by any other method tried.

The author begins by discussing the case of a real dominant root; acceleration by iterating with $A-\mu I$ instead of A; the use of matrix squaring; the Hestenes-Karush iteration; rotational diagonalization of a symmetric matrix; and a method of improving an approximate set of latent vectors. Next he comes to the determination of nondominant roots, first giving methods of removing components of known latent vectors (recommended when the original matrix has many zeros and is best retained intact), and second, methods of reducing the order of the matrix as each root and vector is found. For complex roots and vectors one may obtain the real quadratic satisfied by a pair of conjugate complex roots, thereafter obtaining the corresponding pair of vectors. Otherwise one may iterate with $A-i\mu I$ for suitable μ. The reduction of the matrix raises certain special problems in the complex case which the author deals with.

The theoretical discussion accounts for about two-thirds of the paper, and the rest concerns programming. One series of codes was prepared for dealing with small matrices that can be held in the internal storage; another is used for larger matrices and calls for reading in the matrix from punched cards with each iteration. Double precision products are accumulated and only the sum is rounded. Provision is made for applying Aitken's δ^2-process from time to time when convergence is slow. Provision is made also for adjusting μ and iterating with $A-\mu I$ to accelerate convergence. Careful attention is given to scaling in order to avoid the use of floating point.

The short list of references omits the sources of several of the techniques presented [e.g., the rotational diagonalization discussed by Jacobi, J. Reine Angew. Math. 30, 51–94 (1846); note that the first reference in the list is dated 1952 and should be 1926]. Nevertheless the paper is essentially self-contained, and forms a valuable contribution to the subject.

A. S. Householder (Oak Ridge, Tenn.).

von Neumann, John. A numerical method to determine optimum strategy. Naval Res. Logist. Quart. 1, 109–115 (1954).

The author describes a new numerical iterative process for solving a 2-person zero-sum game given by a rectangular matrix with entries a_{ij}, $i=1,\cdots,m$; $j=1,\cdots,n$. The iteration process will take at most about

$$N=2mn+5(m+n)+(m \text{ or } n)+2$$

microseconds on a machine capable of one multiplication per microsecond. This estimate is lower than the upper bound provided by the simplex method of G. Dantzig. However, the Dantzig method in practice works faster, according to the author, who goes on to state that there appears to be room for substantial acceleration of the new process by means of "various tricks which amount to smoothing the iterative recursive sequence which is involved, and making this recursion dependent on several predecessors." In addition to estimating the length of the iteration, the author estimates the size of the numbers appearing in the successive steps and finds that these remain bounded and therefore suited to machine computation.

J. M. Danskin.

von Mises, Richard. Numerische Berechnung mehrdimensionaler Integrale. Z. Angew. Math. Mech. 34, 201–210 (1954). (English, French and Russian summaries)

The author considers integrals

$$J=\int_V f(x_1,x_2,\cdots,x_k)dx_1dx_2\cdots dx_k=\int_V fdV$$

and their numerical evaluation by formulas of the type $J\approx\sum_{i=1}^n A_if_i$; $f_i=f(P_i)$, where $P_1,P_2,\cdots,P_n$ are points of the domain V and the numbers A_i suitable weights. It is assumed that V is a "star type" domain. This means: An interior point O of V exists, and any straight line leaving O in direction s intersects in exactly one point with the boundary of V. This makes it convenient to introduce polar coordinates r, s with r giving the distance from O. The boundary can be described by a function $r=R(s)=R$. The integral J is rewritten as $J=\int_S[\int_0^R fr^{k-1}dr]ds$, where ds refers to integration over the unit sphere S. It is then assumed that f admits continuous partial derivatives up to order m, and $f^{(t)}$ is to denote the derivative of order t in direction s as assigned to any point $P\neq 0$ by its polar coordinates. It is further assumed that the weights A_i satisfy the conditions

$$\int_V x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_k^{\alpha_k}dV=\sum_{i=1}^n A_ix_{1i}^{\alpha_1}x_{2i}^{\alpha_2}\cdots x_{ki}^{\alpha_k}$$

for all choices of integers $\alpha_j\geqq 0$, $\sum_{j=1}^k\alpha_j\leqq m-1$. Here and in what follows the subscript i refers to point P_i. The main result is the error theorem:

$$D=\int_V fdV-\sum_{i=1}^n A_if_i=\int_S\left[\int_0^R M_i(r)f^{(t)}dr\right]ds$$
$$-\sum_{i=1}^n A_i\int_0^{r_i}N_i(r,r_i)f_i^{(t)}dr;\quad t=1,2,\cdots,m,$$

with

$$M_{i+1}(r)=\frac{1}{i!}\int_0^R u^{k-1}(u-r)^i du$$

$$+(-1)^{i+1}r^{k+i}[k(k+1)\cdots(k+i)]^{-1}$$

and $N_{i+1}(r, r_i)=(i!)^{-1}(r_i-r)^i$. The functions M and N are positive. Due to this fact some upper bounds for $|D|$ are established, one of them being

$$|D|\leq|f^{(k)}|_{\max}\times\frac{1}{i!}\left[\int r^i dV+\sum_{i=1}^{n}|A_i|r_i^i\right].$$

Some examples related to spherical and rectangular domains and especially to the analogue of Simpson's rule conclude the paper. It is remarkable that these basic results are demonstrated in a short and elementary manner.

H. Bückner (Schenectady, N. Y.).

Young, Andrew. Approximate product-integration. Proc. Roy. Soc. London. Ser. A. **224**, 552–561 (1954).

The integral $J=\int_a^b f(x)\Phi(x)dx$ is approximated by a weighted sum $J^*=\sum_{r=1}^{n}p_r f(x_r)$ with suitable weights p_r for prescribed abscissae x_r. The abscissae do not necessarily belong to the interval $a\leqq x\leqq b$. It is assumed that $J=J^*$ for ν linearly independent functions $f=F_0, F_1, \cdots, F_{\nu-1}$, which implies the matrix vector relation $Fp=m$ with $p'=(p_1, p_2, \cdots, p_n)$, $m'=(m_1, m_2, \cdots, m_\nu)$, $m_{\nu+1}=\int_a^b F_\nu\Phi dx$, and with the matrix $F=(F_{ij})$, $F_{ij}=F_{i-1}(x_j)$. In the case $\nu=n$ and for a nonsingular F, the weights are determined by $p=F^{-1}m$; F and F^{-1} do not depend on Φ. Formulas for the weights are given for some significant cases, which include polynomials F_ν and exponential functions $F_\nu=\exp((s-s_0)kx)$ with $k=$const. Special end- and midpoint formulas are also derived for the case of equidistant abscissae. All these formulas involve alternant matrices F, and the paper describes also a desk machine technique (pointed out by A. C. Aitken) for the inversion of alternant matrices. The error of the approximation of the above mentioned integral depends on how closely a function $f(x)$ can be approximated by a linear combination of the functions F_ν. A general formula for the error is also given. A numerical example refers to an integral, where Φ has a singularity for both a and b. It shows that the method is especially useful in such cases.

H. Bückner (Schenectady, N. Y.).

Young, Andrew. The application of approximate product-integration to the numerical solution of integral equations. Proc. Roy. Soc. London. Ser. A. **224**, 561–573 (1954).

Results of the paper reviewed above are applied to both Fredholm and Volterra integral equations. In the case of Fredholm's equation $f(x)+g(x)=\int_a^b K(x, y)f(y)dy$, n abscissae $x_1, x_2, \cdots, x_n$ are chosen, and formulas for the approximate integration of the integrals $\int_a^b\Phi_k(y)f(y)dy$ with $\Phi_k=K(x_k, y)$ are applied. These formulas furnish the exact values of the integrals for the polynomials $f=(y-\eta)^{i-1}$ with $\eta=$const. and $i=1, 2, \cdots, n$. As regards the integral equation, $f(y)$ is approximated by f_k^* in the pivotal point x_k, and the values f_k^* are calculated from the set of linear equations $f_k^*+g(x_k)=\int_a^b K(x_k, y)f^*(y)dy$ for $k=1, 2, \cdots, n$. The integral refers to the polynomial $f^*(y)$ of lowest degree, for which $f^*(x_i)=f_i^*$. This integral is equal to a weighted sum of the values f_i^* according to the integration formula. Some numerical examples are given. These include repeated integration formulas and a method of error throw-back for correction. The examples are compared with results by Fox and Goodwin [Philos. Trans. Roy. Soc. London. Ser. A. **245**, 501–534 (1953); these Rev. **14**, 908] and underline the progress by the method. It is to be noted however that the author's general procedure is not new [see E. J. Nyström, Acta Math. **76**, 157–184 (1945); these Rev. **7**, 220]. For Volterra integral equations $f(x)+g(x)=\int_a^x K(x, y)f(y)dy$ the method is modified. The integration formulas have to furnish the exact values of the integrals $\int_a^{x_k}K(x_k, y)(y-\eta)^{i-1}dy$. They refer to all pivotal points $x_1, \cdots, x_n$. Here again some numerical examples illustrate the usefulness of the method. It would have been interesting, had the author compared his method with A. Huber's procedure [Monatsh. Math. Phys. **47**, 240–246 (1939)] and with the extension of this method as given by C. Wagner [J. Math. Physics **32**, 289–301 (1954); these Rev. **15**, 651]. Huber replaces $f(y)$ by a stepwise linear function, and Wagner extends this to parabolic approximation. *H. Bückner* (Schenectady, N. Y.).

Muggia, Aldo. Sul calcolo dell'integrale di Poisson. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **87**, 116–126 (1953).

Tables are given to facilitate the accurate and rapid calculation of the Poisson-type integral

$$I(\theta_0)=\int f(\theta)\operatorname{ctg}\tfrac{1}{2}(\theta-\theta_0)d\theta,$$

and their use is explained. The major application is to two-dimensional airfoil theory [Theodorsen and Garrick, NACA Rep. no. **452** (1933); I. Flügge-Lotz, NACA Tech. Note no. **2451** (1951); these Rev. **13**, 782]. *G. Birkhoff.*

Zubarev, D. N. Computation of configuration integrals for a system of particles with Coulomb interaction. Doklady Akad. Nauk SSSR (N.S.) **95**, 757–760 (1954). (Russian)

The author discusses

$$Q_N=\int\cdots\int\exp\left[-\frac{1}{2\theta}\sum_{i\neq j}\Phi(r_i-r_j)\right]dr_1\cdots dr_N,$$

where θ is a constant, and $\Phi(r)=\sum_k\nu(k)v^{-1}\exp[i(kr)]$. He obtains an approximation of the form $Q_N=e^{-F/\theta}$, and for Coulomb interaction, $\nu(k)=4\pi e^2/k$ and

$$F=-(2v/3)(\pi/\theta)e^3n^{3/2}.$$

A. Erdélyi (Pasadena, Calif.).

***Zubarev, D. N. Evaluation of configuration integrals for systems of particles with Coulomb interaction.** Morris D. Friedman, Two Pine Street, West Concord, Mass., 1954. 6 pp. (mimeographed). $3.00.

Translation of the paper reviewed above.

Schroeder, J. Zu F. W. Schäfke, Verbesserte Konvergenz- und Fehlerabschätzungen für die Störungsrechnung, Z. angew. Math. Mech. 33 (1953), S. 255–259. Z. Angew. Math. Mech. **34**, 280 (1954).

See these Rev. **15**, 474.

Forsythe, George E. Asymptotic lower bounds for the frequencies of certain polygonal membranes. Pacific J. Math. **4**, 467–480 (1954).

Various approximations for the lowest eigenvalue of the problem $u_{xx}+u_{yy}+\lambda u=0$ for a bounded simply connected open region R, on whose boundary C the function u vanishes, are well known. These approximations, which can be made as good as we please with sufficient labour, are upper bounds

for λ. The paper is concerned with finding an approximation which is a lower bound.

The region R is here supposed to be a convex polygon which is the finite union of squares and half-squares, the side length of each being h. Such a polygon has at most eight vertices, and the angle at a vertex is 45°, 90° or 135°. Evidently if it is the union of squares and half-squares of side h, it is equally the finite union of squares and half-squares of side a submultiple of h.

The vertices of the squares and half-squares are the nodes of a square lattice. The corresponding finite-difference problem

$$v(x+h, y)+v(x-h, y)+v(x, y+h)+v(x, y-h) -4v(x, y)+h^2\lambda_h v(x, y)=0$$

at the interior nodes and $v(x, y)=0$ at the boundary nodes will have a lowest eigenvalue λ_h, and $\lambda_h \to \lambda$ as $h \to 0$. This approximation λ_h is a lower bound for λ for sufficiently small h. *E. T. Copson* (St. Andrews).

Fehlberg, Erwin. Bemerkungen zur numerischen Behandlung des Dirichletschen Problems für allgemeinere Ränder. Acta Math. **91**, 51–74 (1954).

Consider a star-shaped domain D with boundary B; the author proposes to solve the Dirichlet problem for

$$(*) \qquad z_{xx}+z_{yy}=e(x, y)$$

in D as follows: enclose D inside an ellipse E whose boundary is as close as possible to B; map E onto the unit circle; then project the boundary values of z (prescribed on the image B' of B) onto the unit circle; obtain a particular solution of (*) by expanding z and e in double-series of Legendre polynomials and imposing the "projected" boundary-conditions at the unit circle; add a correction (satisfying Laplace's equation) which best fits the "residual" boundary conditions along B', in the sense of least squares. Extensive formulas and a numerical example are included. The technique can be extended to more general elliptic equations using Picard's method of successive approximations. [For similar ideas, but an entirely different procedure, see Hyman, Appl. Sci. Research B. **2**, 325–351 (1952); these Rev. **13**, 993.]

M. A. Hyman (Pittsburgh, Pa.).

Volkov, E. A. On a method of increasing the accuracy of the method of grids. Doklady Akad. Nauk SSSR (N.S.) **96**, 685–688 (1954). (Russian)

This paper treats certain methods proposed by L. Fox [Proc. Roy. Soc. London. Ser. A. **190**, 31–59 (1947); these Rev. **9**, 106] for refining the results of relaxation methods applied to Poisson's equation with a square mesh. Noting that it is not always clear in these methods whether convergence takes place, the author proceeds to a critical examination of a special problem, i.e., the case when a curvilinear boundary cuts the mesh at other than nodal points, and where an interpolation formula is used to estimate values at nodal points adjacent to the boundary. For this case sufficient conditions for convergence are supplied.

W. E. Milne (Corvallis, Ore.).

Cornock, A. F. The numerical solution of Poisson's and the bi-harmonic equations by matrices. Proc. Cambridge Philos. Soc. **50**, 524–535 (1954).

The author describes a direct method of solving the linear equations obtained from the customary finite-difference approximations for the first boundary problem for the Poisson and the bi-harmonic equation (in two and three dimensions). His approach makes use of the fact that the matrix M of coefficients of these difference equations has a simple partitioned form if the region in which the solution is sought can be subdivided into a few rectangular areas (or rectangular parallelepipeds).

The technique consists in premultiplying the matrix M by a suitable matrix so that the resulting matrix M_1 has a partitioned form which is lower triangular. Furthermore, all of the submatrices on the principal diagonal of M_1 are unit matrices, except for the submatrix in the first position (which is to be non-singular). Therefore for a square region in two dimensions, he need only solve by elimination a system of n linear equations (instead of n^2), where n is the number of net points in each direction. The rest of the work necessary to obtain the solution of the original n^2 equations involves matrix multiplication of matrices of order $n \times n$. In the case of a rectangular parallelepiped he similarly reduces the problem to the inversion of a single matrix of order n^2 together with the multiplication of matrices of order n^2. He remarks that for regions with less regular boundaries the present method doesn't seem practicable.

E. Isaacson (New York, N. Y.).

Platzman, G. W. The computational stability of boundary conditions in numerical integration of the vorticity equation. Arch. Meteorol. Geophys. Bioklimatol. Ser. A. **7**, 29–40 (1954).

In the numerical forecasting of flow patterns in the atmosphere the region over which the integration is performed is limited artificially by a boundary introduced to keep the total number of data to a reasonable size. The introduction of such a boundary requires the prescription of conditions along this curve. These conditions are not necessarily in conformance with the physical model and may therefore introduce errors in the calculation. The author considers a particular finite-difference formulation of the problem and three sets of boundary conditions. He discusses the numerical stability of the problem under these different sets of conditions. *H. H. Goldstine* (Princeton, N. J.).

Heinhold, J., und Albrecht, R. Zur Praxis der konformen Abbildung. Rend. Circ. Mat. Palermo (2) **3**, 130–148 (1954).

For the approximation of the function which maps a given region conformally on the interior of a circle, it is suggested that a combination of several methods be used, especially for regions which are not nearly circular. For such regions it is suggested that the Schmiegungsverfahren be used to reduce the problem to the problem of mapping of a nearly circular region, which in turn may be handled by the Theodorsen-Garrick method. Functions which are both efficient and easy to handle are proposed for the Schmiegungsverfahren. These include some derived from mappings of circles with spines. Detailed formulas for numerical calculation are given. *C. Saltzer* (Cleveland, Ohio).

Clippinger, R. F., Dimsdale, B., and Levin, J. H. Automatic digital computers in industrial research. IV. J. Soc. Indust. Appl. Math. **2**, 113–131 (1954).

For part III see same J. **2**, 36–56 (1954); these Rev. **15**, 901.

Mitrović, Dusan. Sur un principe nouveau de construction des machines électriques destinées à résoudre les équations algébriques. Ann. Fac. Sci. Univ. Toulouse (4) **16** (1952), 204–211 (1953).

Mitrović, Dusan. Etude théorique d'un principe nouveau de construction des machines électriques servant à résoudre les systèmes d'équations algébriques linéaires. Ann. Fac. Sci. Univ. Toulouse (4) **16** (1952), 169–203 (1953).

Tea, Peter L. A string-net analog for the numerical solution of the equations of Laplace and Poisson. J. Franklin Inst. **258**, 287–303 (1954).

Medgyessy, Pál. Un appareil nouveau pour la synthèse et analyse de Fourier ainsi que pour des problèmes analogues. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **2** (1953), 179–185 (1954). (Hungarian. Russian and French summaries)

Fazekas, Ferenc. Mathematische Untersuchung der Genauigkeit einer Ungarischen Kopiereinrichtung. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **2** (1953), 415–446 (1954). (Hungarian. Russian and German summaries)

Török, Vilmos. An investigation of the copying lathe "Hydrofix" by the theory of servomechanisms. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **2** (1953), 447–455 (1954). (Hungarian. Russian and English summaries)

Sgarbazzini, Carlo. Visione sintetica della "nomografia." Archimede **6**, 150–155 (1954).

ASTRONOMY

Vernić, Radovan. Periodische Lösungen im Dreikörperproblem. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **8**, 247–266 (1953). (Serbo-Croatian summary)

By means of the Sundman-Levi-Civita transformation $dt=du/V$ the Lagrange equality assumes the form

$$d(VdJ/du)/du=2(1+2h/V), \tag{1}$$

which the author has chosen as his working equation. In (1) V denotes the potential of the system of three particles m_i $(i=1, 2, 3)$, J is the Lagrange function (angular momentum) and h is the constant of total energy. The main results of the first part of the paper are as follows. (i) For the periodic solutions of the general three-body problem the Lagrange function J and the potential V are periodic functions with one and the same period. (ii) The periodic solutions of the general three-body problem are to be found among the periodic solutions of the equation (1). (iii) A necessary condition for the existence of periodic solutions in the general three-body problem is that $h<0$.

In the second part of the paper use is made of Green's function for solving the boundary-value problem of periodicity for an ordinary linear second-order self-adjoint differential equation. Equation (1) being of parabolic type, it is shown that this method of constructing periodic solutions of (1) gives all possible periodic solutions. The third part of the paper consists of a single theorem which states that the only periodic solutions of the general three-body problem are the Lagrange exact solutions.

By the results of this paper and several other of his papers which have been inaccessible to the reviewer, the author claims to have completely solved the Poincaré problem in the three-body problem, namely, the problem of existence of periodic solutions. The reviewer believes that this claim will cause some controversy. *E. Leimanis.*

Grémillard, Jean. Sur les racines d'une équation de la théorie des solutions périodiques de la troisième sorte. C. R. Acad. Sci. Paris **239**, 153–155 (1954).

The author improves the proof of the following theorem of von Zeipel [Nova Acta Soc. Sci. Upsaliensis. Sect. I. (3) **20**, no. 9 (1904)] concerning the initial value of the mean longitudes which correspond to periodic solutions of the third kind (sorte) in the three-body problem. "The equation $\partial R_0/\partial\bar{\lambda}=0$ has no other solutions for $\bar{\lambda}$ than those given by $\bar{\lambda}=\frac{1}{2}r\pi$ if $p-q$ is odd and $\bar{\lambda}=r\pi$ if $p-q$ is even, where $r=0, \pm1, \pm2, \cdots$." R_0 denotes that part of the mean value of the perturbative function which is independent of the eccentricities, λ and λ' are the mean longitudes of the two planets, and $\bar{\lambda}=p\lambda_0'-q\lambda_0$, p, q being relatively prime positive integers [cf. Grémillard, same C. R. **234**, 2339–2341 (1952); **236**, 49–51 (1953); these Rev. **13**, 996; **14**, 802]. *E. Leimanis.*

Armellini, Giuseppe. Sopra le variazioni dell'eccentricità nel problema astronomico dei due corpi di masse decrescenti. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **15** (1953), 345–351 (1954).

The author remarks that if a star increases in mass and this increase takes place isotropically the equation of motion for this star is that of Levi-Civita, $d(m\mathbf{V})/dt=\mathbf{F}$. If the star decreases in mass, also isotropically, the equation of motion is that of Gyldén, $m d\mathbf{V}/dt=\mathbf{F}$. Using the last equation he proves that in the case of a double-star system, if the total mass decrease takes place so that

$$\lim_{t\to\infty} M(t)=\lim_{t\to\infty} tM(t)=0, \tag{1}$$

then both the eccentricity and the radius vector will grow above any arbitrary limit. The orbit of one of the stars with respect to the other will change from an ellipse into a parabola and then into a hyperbola. Besides the mass decrease by light radiation a star also loses mass by corpuscular radiation. Both will be proportional to the mass so that $dM/dt=-\epsilon M$ and $M(t)=M(0)\exp(-\epsilon t)$, and condition (1) is satisfied. However, in most cases also a mass accretion of the star will take place, for which Levi-Civita's equation must be used. The author intends to deal with this case in a following paper. *A. J. J. van Woerkom.*

Rasmusen, H. Q., and Hesselberg, O. K. An appropriate method for integration of the motion of periodic comets. Danske Vid. Selsk. Mat.-Fys. Medd. **28**, no. 10, 40 pp. (1954).

In the application of the method of numerical integration of the perturbed motion of a planet or comet, the individual attractions of the four inner planets may usually be ignored if the perturbed body is at a distance more than four or five astronomical units from the sun. At smaller distances, the mass of Mercury is almost always included in that of the Sun, but the attractions by Venus, the earth and Mars are taken rigorously into account. Their rapid motions make an interval exceeding 10 or 20 days impossible. For objects moving in highly elliptical orbits the usual practice is to use both of these methods alternately, but at the transition from one method to the other discontinuities may be introduced.

To overcome this difficulty, the authors recommend using a coordinate system having its origin at the center of mass of the sun and the four inner planets. Its application requires a table of the coordinates of the sun in this system. The paper furnishes such a table, with the sign reversed, to the tenth decimal in astronomical units with interval 10 days for the years 1920–1960. A collection of formulas and a numerical example are also given. *D. Brouwer.*

Musen, Peter. Special perturbations of the vectorial elements. Astr. J. 59, 262–267 (1954).

Vectorial methods are becoming increasingly popular in mechanics. This paper deals with the theory of special perturbations in the vectorial elements. If (P, Q, R) is an orthogonal set of unit vectors with P in the direction of the pericenter and R at right angles to the orbital plane, the following vectorial elements have been used

$$\mathbf{a}=e\mathbf{P},\quad \mathbf{b}=ep^{\frac{1}{2}}\mathbf{Q},\quad c=p^{\frac{1}{2}}\mathbf{R},\quad \mathbf{g}=\frac{e}{p^{\frac{1}{2}}}\mathbf{Q}.$$

Any two of these vectors, except b and g are equivalent to five scalar elements. Differential equations are derived for these vectorial elements using Brown's differential operator δ/dt. In this operator δ is the variation caused by the disturbing force only. The author shows that the equations are derived in a very elegant way by applying this δ/dt operator to the integral of areas and the Laplacian and Hamiltonian integrals. By simple vectorial transformations the equations for the elliptic elements can be found. The author proposes a new system of equations that has certain advantages over the systems of Strömgren and Herrick. The appearance of the eccentricity in the denominator is very troublesome in the equations for the variation of the orbital elements. It appears in the equations for a, b and g. By using c and g the new system is superior to Herrick's who uses a and b. It is also superior to Strömgren's because of the use of a fixed coordinate system. The author concludes his paper with a sample computation which includes checks that can be applied very readily. Of all the papers on this subject this is the most comprehensive and elegant.

A. J. J. van Woerkom (New Haven, Conn.).

Kopal, Zdeněk. The propagation of shock waves in self-gravitating gas spheres. Astrophys. J. 120, 159–171 (1954).

The equations of motion of unsteady spherically symmetrical flow of a compressible, inviscid, self-gravitating sphere of gas with constant specific heats (in the ratio γ) are written down. Solutions are sought in which ut/r, pt^4/r^3 and ρt^2 are functions of $\xi=t^\psi/r$ only. Here ψ is a constant, r is distance from the centre, t is the time, u the radial velocity, p the pressure and ρ the density. For any values of ψ and γ the equations of motion reduce to a system of ordinary differential equations. The flow is compatible with being separated by a spherical shock wave, on which ξ takes any constant value ξ_1, from gas, outside the shock wave, which is in its initial equilibrium state with the density proportional to $r^{-\alpha}$, provided that $\psi=2/\alpha$ and $1<\alpha<3$. Different values of ξ_1 correspond to different constant values of the Mach number of the shock wave. One may note also that the interior of a second sphere, on which ξ is another constant ξ_2, is empty; particles initially at the centre move outward according to the law $r=t^\psi/\xi_2$.

The question now arises whether all these solutions have physical reality. In two previous papers [Carrus, Fox, Haas, and Kopal, Astrophys. J. 113, 193–209, 496–518 (1951); these Rev. 12, 643; 13, 168] this was investigated by laying down as a condition that the total energy (kinetic, thermal and gravitational) interior to the shock wave remains constant in time. This was found to occur for all shock Mach numbers if $\alpha=2.5$, but not at all for other values of α. In the present paper it is pointed out that in practice the energy inside the shock wave will increase continuously by absorbing the thermal and gravitational energy of the undisturbed gas through which it passes. Hence the earlier work is applicable only to stellar explosions in which an amount of energy is released which is very large compared with the total thermal and gravitational energy of the star. The other extreme case, in which it is very small by comparison, is nearer to the conditions in novae, and is that treated in the present paper. The total energy inside the shock wave is taken equal to the energy inside the same sphere in the initial undisturbed state. It is found that a solution satisfying this condition exists for each value of $\alpha<2.5$, but for only one value of ξ_1, and hence only one shock Mach number (which tends to ∞ as $\alpha\to 2.5$), for each α. Numerical details of seven cases (with M between $\sqrt{3}$ and $\sqrt{500}$) are given.

The physical discussion of this result is necessarily a bit inconclusive, as the possibility of flows headed by shocks of other strengths but without any similarity of profile at different times (which the author calls the "progressing wave" condition) is not ruled out. A not impossible inference, however, that for a given density law (with α between 2.5 and some lower limit) in the undisturbed state, any weak enough explosion will generate a shock wave, which will tend to a certain fixed Mach number and remain at it.

M. J. Lighthill (Manchester).

Mackie, A. G. Expansion of a finite one-dimensional gas cloud into a vacuum. Proc. Roy. Soc. Edinburgh. Sect. A. 64, 57–70 (1954).

The paper deals with the adiabatic expansion of a one-dimensional finite gas cloud which is bounded on both sides by a vacuum. The author takes the case where the ratio of the specific heats of the gas, γ, is 5/3, the value for monatomic hydrogen. Initially the gas cloud is at rest but is allowed to expand under the action of its pressure gradient. An initial density distribution is assumed in which the density is continuous and rises steadily from zero at the boundaries $x=h_1$, $x=-h_2$ to a maximum at $x=0$ in the interior of the cloud. The motion is assumed to be continuous and an analytic solution is found in each of seven regions in the (x,t)-plane by using the Riemann invariants $r=\frac{1}{2}(3c+u)$, $s=\frac{1}{2}(3c-u)$, where c is the local velocity of sound and u is the velocity of the gas. The method of characteristics is then used to determine the region of applicability of each solution and also the lines across which the different solutions must be fitted. The motion of one of the boundaries is investigated and this is found to depend solely on the values of the derivative of the initial density distributions at the boundary in question, and at the position of maximum density, $x=0$. Conditions are found for the boundary to have (i) an initial stationary period and (ii) a final constant velocity of advance into the vacuum. Finally the author considers the problem of the breakdown of the flow and obtains a rather restrictive condition on the initial density distribution for the flow to remain continuous.

M. H. Rogers (Urbana, Ill.).

McVittie, G. C. Aerodynamic motions of interstellar gas clouds. Astrophys. J. 119, 352–364 (1954).

In continuation of his earlier investigation [Quart. Appl. Math. 11, 327–336 (1953); these Rev. 15, 175] the author develops a new method for solving the equations of motion of compressible gas flow, which are deduced from the Newtonian approximation to the equations of general relativity, for the case of one-dimensional motion. The resulting equations take account of gravitational self-attraction and of pressure-gradient in the gas. In the light of such equations the author considers the motion of a thin self-gravitating gas cloud bounded by two parallel plane shocks and moving normally to them in vacuum (or a gas at much lower density). It is shown that when the velocity (or acceleration) is constant, adiabatic motion is possible; but when the acceleration varies the time, the cloud may be losing energy by radiation. The width of the cloud in the direction of its motion is calculated in terms of the density and temperature of its densest portion; and applications to interstellar clouds of 0.3–45 parsecs in size are carried out.

Z. Kopal (Manchester).

Sestini, Giorgio. Sulla regolarità dei moti unidimensionali di un mezzo continuo disgregato. Rivista Mat. Univ. Parma 4, 241–254 (1953).

Levi-Civita [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 2, 365–371 (1925)] derived conditions for a one-dimensional self-gravitating medium (all motion perpendicular to a certain plane) failing to be regular, i.e. density becoming infinite. The present paper treats the same problem for two other one-dimensional cases: spherical symmetry and axial symmetry. In each case there is assumed present a rigid nucleus, a sphere and right circular cylinder, respectively, for which internal forces are neglected.

R. G. Langebartel (Urbana, Ill.).

Gjellestad, Guro. On the elongation of a fluid sphere in a uniform external magnetic field. Astrophys. J. 120, 172–177 (1954).

A nonmagnetic homogeneous incompressible fluid sphere of infinite electrical conductivity in a uniform magnetic field is subjected to the deformation $\epsilon P_n(\mu)$ where $P_n(\mu)$ is the Legendre polynomial of order n. The change in the magnetic energy is computed to the first order in ϵ, and for the case $n=2$ the result indicates a tendency toward a prolate spheroidal shape. The magnetic energy change for other values of n is of a higher order and therefore not obtained in this first-order theory.

R. G. Langebartel.

Ferraro, V. C. A. On the equilibrium of magnetic stars. Astrophys. J. 119, 407–412 (1954).

As was previously pointed out by Chandrasekhar and Fermi [Astrophys. J. 118, 116–141 (1953); these Rev. 15, 168], the magnetic field which is uniform inside a liquid sphere and outside behaves like a dipole tends to flatten the sphere in a manner similar to the effects of axial rotation, so that its eccentricity $e=3.5M/W$, where M denotes the magnetic energy, and W the gravitational energy, of the sphere. The author of the present investigation considers the equilibrium of a self-gravitating liquid sphere endowed with slow rotation, in which electric currents flow symmetrically about the axis of rotation. He shows that the magnetic field must fulfil certain conditions (a fuller discussion of which is postponed for further communication) if equilibrium is to prevail. Provided that the magnetic energy of the currents is small in comparison with the gravitational energy, the free surface of the liquid will become a slightly flattened spheroid of eccentricity

$$e^2=\frac{15\omega^2}{8\pi G\rho}+\frac{225}{64}\frac{H_p^2}{\pi^2 G\rho^2 a^2},$$

where ω denotes the angular velocity of axial rotation; ρ, the (constant) density of the liquid; H_p, the magnetic-field intensity; G, the constant of gravitation; and a, the mean radius of the configuration. An application of this formula to practical cases reveals that magnetic fields known to exist in certain classes of stars are apt to produce barely perceptible effects.

Z. Kopal (Manchester).

Lyttkens, Ejnar. On the radial pulsations of an infinite cylinder with a magnetic field parallel to its axis. Astrophys. J. 119, 413–424 (1954).

The aim of the present investigation has been to determine the free periods of radial adiabatic oscillations of an infinitely conducting and infinitely long cylinder under the influence of its own gravity and of a magnetic field parallel with its axis. In sec. I the author deduces the particular form of Chandrasekhar and Fermi's equation [Astrophys. J. 118, 116–141 (1953); these Rev. 15, 168] governing small oscillations of a homogeneous cylinder, together with the associated boundary conditions (requiring that the amplitude vanishes on the axis of the cylinder, and remains finite on the surface). Sec. II deals with an analytical determination of characteristic functions of the problem in the absence of a magnetic field. In sec. III the magnetic field is assumed to be proportional to the square-root of the pressure (in which case the characteristic functions turn out to be the same as if the field were absent); and in sec. IV the author considers the effects of a slow secular decay of the field. In the concluding sec. V the author sets out to construct more general solutions of his problem by approximate variational methods, and to bracket the characteristic frequencies of radial oscillations with the aid of Rayleigh's principle (giving the upper limit) and of Temple's inequality (furnishing the lower limit). Numerical values of such characteristic periods for certain types of magnetic fields are given.

Z. Kopal (Manchester).

Chatterji, L. D. Radial pulsations of the two stellar models. Proc. Nat. Acad. Sci. India. Sect. A. 21, 118–125 (1952).

The period of the fundamental mode of radial oscillations is obtained for the two models with the density laws $1-r/R$ and $1-r^2/R^2$.

R. G. Langelbartel (Urbana, Ill.).

Chatterji, L. D. Radial oscillations of a polytropic model of index 4. Proc. Nat. Acad. Sci. India. Sect. A. 21, 268–272 (1952).

Following the numerical integration method of his earlier paper [Proc. Nat. Inst. Sci. India 17, 467–470 (1951); these Rev. 14, 93] the author computes the periods of the fundamental mode and first overtone for small radial adiabatic oscillations for a gaseous star with polytropic index four.

R. G. Langebartel (Urbana, Ill.).

Kushwaha, R. S. Stability of higher modes for stellar models under variable Γ. Proc. Nat. Acad. Sci. India. Sect. A. 20, 229–238 (1951).

From a study of four different models ranging from the homogeneous to the Roche model with $\Gamma=\Gamma_0(1-kr^2/R^2)$ and $\Gamma=\Gamma_0(1-kr^4/R^4)$ the author concludes that the funda-

mental and higher modes of adiabatic radial oscillations for these models are stable. *R. G. Langebartel.*

Kushwaha, R. S. The effect of amplitude and higher modes on the period and skewness for anharmonic oscillations of Roche-model. Proc. Nat. Acad. Sci. India. Sect. A. **20**, 239–242 (1951).

The author finds that on the basis of the anharmonic oscillation theory for the Roche model with $\gamma=5/3$ the effect of the higher modes and increased amplitude is insufficient to explain the observed skewness.

R. G. Langebartel (Urbana, Ill.).

Khare, R. C. Instability of non-radial oscillations of centrally condensed stars. Proc. Nat. Inst. Sci. India **20**, 326–335 (1954).

Non-radial oscillations in the Roche model are studied for the special case where the specific heat ratio is 1.5. The amplitude of the oscillations is shown to be infinite at the surface of the star. This instability is suggested as the explanation for the origin of the planetary nebulae rather than the nova-explosion theory. This view is supported by the observation that the number of nova-explosions is large compared to the number of planetary nebulae.

R. G. Langebartel (Urbana, Ill.).

Sobolev, V. V. On the theory of an unsteady radiation field. I. Akad. Nauk SSSR. Astr. Žurnal **29**, 406–417 (1952). (Russian)

The radiation from a continuously varying light source is modified in a dispersive medium. The one-dimensional problem is studied by considering the probability of escape from the medium of a quantum of light absorbed at a given optical depth at a given instant. A third-order linear partial differential equation satisfied by this quantity is derived and solved by the separation of variables method for the two cases of finite and infinite optical thickness. The results are used to compute the mean time of stay of a quantum in the medium. Novae envelopes and solar prominences are suggested as fields for applications. *R. G. Langebartel.*

Sedov, L. I. On theoretical formulas for the stellar laws "mass-luminosity" and "mass-radius." Doklady Akad. Nauk SSSR (N.S.) **94**, 643–646 (1954). (Russian)

Under somewhat more general assumptions regarding the laws satisfied by the rate of generation of energy and the coefficient of opacity than those adopted by Strömgren [Handbuch der Astrophysik, Bd 7, Springer, Berlin, 1936, p. 159] a corresponding mass-luminosity-radius formula is derived by means of a dimensional analysis.

R. G. Langebartel (Urbana, Ill.).

RELATIVITY

Hély, Jean. Sur la représentation d'Einstein du champ unitaire. C. R. Acad. Sci. Paris **239**, 385–387 (1954).

The author defines the tensor

$$\sqrt{(-f)}f^{ij}=\sqrt{(-g)}g^{ij},$$

where $f^{-1}=\det\|f^{ij}\|$, $g^{-1}=\det\|g^{ij}\|$ and g^{ij} is the symmetric part of g^{ij}, the non-symmetrical fundamental tensor of Einstein's generalized theory of relativity. He then gives arguments for interpreting the tensor f^{ij} as that associated with the gravitational field. *A. H. Taub* (Urbana, Ill.).

Vaidya, P. C. Spherically symmetric solutions in non-symmetrical field theories. II. Physical Rev. (2) **96**, 5–9 (1954).

The author corrects his previous derivation [Physical Rev. (2) **90**, 695–698 (1953); these Rev. **15**, 168] of a non-symmetric space-time tensor g_{ik} which is to be spherically symmetric in 3-space. He finds that in the unified field theories of Einstein [The meaning of relativity, 4th ed., Princeton, 1953, Appendix II; these Rev. **14**, 805], Einstein and Straus [Ann. of Math. (2) **47**, 731–741 (1946); these Rev. **8**, 412], and Schrödinger [Proc. Roy. Irish Acad. Sect. A. **51**, 163–171 (1947); these Rev. **9**, 310], conditions imposed by the field equations restrict the form of his tensor to that derived previously by Papapetrou [ibid. **52**, 69–86 (1948); these Rev. **10**, 580]. *F. A. E. Pirani* (Dublin).

Yano, Kentaro, and Ohgane, Masayoshi. On six-dimensional unified field theories. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **13**, 99–132 (1954).

In a previous paper it was proved that all geometries appearing in five-dimensional unified field theory are special cases of the geometry of a V_5 (5-dim. Riemannian space) in which a vector field is given. In this paper the same is proved for the geometries of all six-dimensional unified field theories with respect to a V_6 in which a 2-parameter group of transformations is given. Starting from a complete system of two partial differential equations in V_{n+2} an anholonomic coordinate system is established and the anholonomity object is computed. By means of this a V^n_{n+2} can be defined and with this V^n_{n+2} the properties are considered that are invariant for the transformations of the 2-parameter group. In §4 non-holonomic geodesics are discussed as possible trajectories of test particles. Curvature tensors and field equations are dealt with in §5. In §6 a number of six-dimensional unified field theories are deduced as special cases of the more general conceptions of this paper.

J. A. Schouten (Epe).

Takasu, Tsurusaburo. A necessary unitary field theory as a non-holonomic parabolic Lie geometry realized in the three-dimensional Cartesian space. Proc. Japan Acad. **29**, 533–536 (1953).

This is a short publication of results dealt with in greater detail in the paper reviewed below. *J. A. Schouten.*

Takasu, Tsurusaburo. A necessary unitary field theory as a non-holonomic parabolic Lie geometry realized in the three-dimensional Cartesian space and its quantum mechanics. Yokohama Math. J. **1**, 263–273 (1953).

Einstein's generalized unitary field theory is brought into connection with a non-holonomic parabolic Lie geometry realized in a three-dimensional cartesian space. Generalizations are given of the Dirac equation and of the Schrödinger equation. A short summary of results was published in the paper reviewed above. *J. A. Schouten* (Epe).

Pirani, F. A. E. On the influence of the expansion of space on the gravitational field surrounding an isolated body. Proc. Cambridge Philos. Soc. **50**, 637–638 (1954).

The author disputes the statement by Einstein and Straus [Rev. Modern Physics **17**, 12–124 (1945); these Rev. **7**, 87] that the expansion of the universe can be neglected in considering the gravitational field surrounding a local con-

densation. Applying the jump conditions at discontinuities in general relativity [O'Brien and Synge, Communications Dublin Inst. Advanced Studies. Ser. A. no. 9 (1952); these Rev. 14, 913] to the field equations with cosmical constant λ, he concludes that λ must be continuous across a spherical boundary B, outside which $\rho \neq 0$ and the metric is homogeneous and isotropic, and inside which $\rho = 0$, except possibly at or near the space-origin, and the metric is spherically symmetric. Under these conditions it follows that the expansion of space does influence the structure of the gravitational field surrounding an isolated body. It is shown further that inside B there is a finite maximum distance for a particle if it is not to spiral indefinitely outwards. A simple calculation, with numerical values appropriate to the local cluster of nebulae, indicates that this critical distance is of the same order of magnitude as the observed radius of the cluster, although it appears that the author has taken too high a value for Hubble's constant. *G. J. Whitrow* (London).

Salzman, G., and Taub, A. H. Born-type rigid motion in relativity. Physical Rev. (2) **95**, 1659–1669 (1954).

The results of Born, Herglotz and Noether on the classical motion of rigid systems of particles are first discussed in the special and general theory of relativity and a derivation of the differential equations of the Born-type rigid motion is given. These equations are expressed in a general covariant form by the introduction of an arbitrary metric tensor. The authors then discuss the problem of the approximate solution of the field equations of general relativity, the equations of motion of a perfect fluid undergoing a rigid motion, and the boundary condition $p = 0$, where p is the invariant pressure. It is shown to the first approximation that there exist in the classical limit of the theory rotating dynamical systems held together by their own gravitational fields.

G. Stephenson (London).

Taub, A. H. General relativistic variational principle for perfect fluids. Physical Rev. (2) **94**, 1468–1470 (1954).

The author accepts the usual field equations (A) $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\kappa T_{\mu\nu}$ and takes for a perfect fluid (with no viscosity or heat conductivity) (B) $T^{\mu\nu} = \rho\lambda u^\mu u^\nu - pg^{\mu\nu}$. [He uses a zero suffix on ρ and on some other symbols; these seem unnecessary and are omitted for simplicity in this review.] Here ρ is the rest density of the fluid (i.e. the number of particles per unit volume as measured by an observer moving with the mean velocity of these particles), p is the pressure, $\lambda = c^2 + \epsilon + p/\rho$, where ϵ is the specific internal energy of the fluid in the rest frame; u^μ is the 4-velocity, so that (C) $g_{\mu\nu}u^\mu u^\nu = 1$. The equation of the conservation of mass (D) $(\rho u^\nu)_{;\nu} = 0$ is accepted. [ρ as defined above has the wrong physical dimensions; the author must mean total proper mass per unit volume.] The entropy equation

$$TdS = d\epsilon + pd(1/\rho)$$

is accepted; ρ, T are taken as independent thermodynamic variables, an equation of state being implied. If $g_{\mu\nu}$ are given, we have in $T^{\mu\nu}{}_{;\nu} = 0$ and (C, D) 6 equations for the 6 quantities u^μ, ρ, T, and so the motion is determined by the field. [Since the field is due to the fluid, it might be better to say that (A, C, D) and 4 coordinate conditions are 16 equations for the 16 quantities $g_{\mu\nu}$, u^μ, ρ, T.] Generalising his earlier nonrelativistic variational principle [Proc. Symposia Appl. Math., Vol. I, Amer. Math. Soc., New York, 1949, pp. 148–157; these Rev. 11, 222], the author takes the integral

$$I = \int [R - 2\kappa\rho(c^2 + H + \tfrac{1}{2}\mu g_{\mu\nu}u^\mu u^\nu)]\sqrt{(-g)}d^4x,$$

where $H = \epsilon - TS$ and μ a Lagrange multiplier. The variational principle $\delta I = 0$ (with μ unvaried) is subject to the condition that (D) is always satisfied and μ is to be found so that (C) is satisfied in the stationary state. The principle yields the field equations (A) with $T^{\mu\nu}$ as in (B), and also the constancy of S along a world line. In the variation all field quantities are varied. The essential technique is to express the world lines in the form $x^\mu = x^\mu(u, v, w, s)$ where u, v, w are variables labelling a particular world line and s is proper time on it, and transform to the coordinates (u, v, w, s). [The reviewer finds difficulty in understanding what picture of a fluid underlies this work. In a macroscopic (phenomenological) picture there are no particles which can be counted and (D) is merely an ad hoc assumption; moreover, the separation of ϵ from ρ seems out of keeping with the usual relativistic identification of energy and mass. In a microscopic picture the particles may have either a common motion or random motions; if the former, then (D) is natural, but we would be inclined to put $\epsilon = p = 0$ in the case of a gas, since energy and pressure come from random motions; if the latter, the mean velocity requires definition, and (D) is again an ad hoc assumption. These remarks do not of course reflect on the mathematical validity of the deduction of the differential equations from the variational principle.]

J. L. Synge (Dublin).

Kluitenberg, G. A., de Groot, S. R., and Mazur, P. Relativistic thermodynamics of irreversible processes. II. Heat conduction and diffusion; physical part. Physica **19**, 1079–1094 (1953).

In Part I of this series of papers [Physica **19**, 689–704 (1953); these Rev. **15**, 490] the authors described the flow of a chemical component j by means of a 4-vector $I_\alpha^{(j)}$ orthogonal to the barycentric 4-velocity. They now reformulate their work, using instead the 3-vector

$$\mathbf{J}^{(j)} = \rho^{(j)}(\mathbf{v}^{(j)} - \mathbf{v}),$$

where $\rho^{(j)}$ is the relative density of proper mass, $\mathbf{v}^{(j)}$ the 3-velocity of component j, and $\mathbf{v}$ the barycentric 3-velocity. In the reformulation, specific entropy appears as the fourth component of a 4-vector, to within a factor. [The basis of this paper is the same as that of Part I, and on reexamining that basis the reviewer is of the opinion that the theory is either (a) physically uninteresting or (b) mathematically incorrect. The theory deals with a mixture of n chemical components. Each component j has a density $\rho^{(j)}$ (sum of proper masses of all j-particles in unit volume) and a 3-velocity $\mathbf{v}^{(j)}$; the authors assert that the four quantities

$$m_1^{(j)} = \rho^{(j)}v_1^{(j)}, \quad m_2^{(j)} = \rho^{(j)}v_2^{(j)}, \quad m_3^{(j)} = \rho^{(j)}v_3^{(j)}, \quad m_4^{(j)} = ic\rho^{(j)},$$

are the components of a 4-vector. They state that $\mathbf{v}^{(j)}$ is the velocity of component j. If this means that all j-particles have the same velocity $\mathbf{v}^{(j)}$, then the statement that $m_\alpha^{(j)}$ is a 4-vector is mathematically correct. But the physical picture is trivial; for a simple gas ($n = 1$), all the particles have the same velocity, i.e. the absolute temperature is zero. In view of the title of these papers one must assume that the authors had a different intention. If one takes a realistic view of the system, assigning a general distribution of

velocities to the j-particles, then $\mathbf{v}^{(j)}$ must be some sort of mean velocity for the component j. But however $\mathbf{v}^{(j)}$ is defined, the quantities $m_\alpha{}^{(j)}$ as written above cannot be the components of a 4-vector, because $\rho^{(j)}$ is made up of contributions from particles with different velocities, and the transformation of each such contribution depends on the individual velocity. It appears to the reviewer that the theory can be rescued from this impasse in two ways. Either the definition of $\rho^{(j)}$ must be abandoned and $\rho^{(j)}$, $\mathbf{v}^{(j)}$ accepted as macroscopic quantities without microscopic interpretation, or the energy tensor of the j-particles must be constructed and $\rho^{(j)}$, $\mathbf{v}^{(j)}$ defined in terms of its timelike eigenfunction and eigenvector [cf. J. L. Synge, Trans. Roy. Soc. Canada. Sect. III. (3) **28**, 127–171 (1934)].]

J. L. Synge (Dublin).

Kluitenberg, G. A., and de Groot, S. R. Relativistic thermodynamics of irreversible processes. III. Systems without polarization and magnetization in an electromagnetic field. Physica **20**, 199–209 (1954).

Continuing previous work (see preceding review), the authors study the effects of an electromagnetic field on a system of charged particles, the field acting on the particles according to the usual Lorentz law of ponderomotive force. [The criticism given in the preceding review applies here also. By inserting in the Lorentz formula the $\mathbf{v}^{(k)}$ for chemical component k, the authors implicitly confirm their statement that each component has only one velocity instead of a distribution of velocities. It appears to the reviewer that the formal manipulations of this paper contribute little to the subject indicated in the title.] *J. L. Synge.*

MECHANICS

Simon, Herbert A. The axiomatization of classical mechanics. Philos. Sci. **21**, 340–343 (1954).

Discussion of Simon, Philos. Mag. (7) **38**, 888–905 (1947) [these Rev. **10**, 71] and McKinsey, Sugar and Suppes, J. Rational Mech. Anal. **2**, 253–272 (1953) [these Rev. **14**, 1023].

***Bouligand, Georges. Mécanique rationnelle. Cours et problèmes résolus à l'usage des élèves des facultés des sciences.** 5ème éd. Librairie Vuibert, Paris, 1954. xxxii +572 pp. 2400 francs.

This is a course on mechanics in the French tradition, which changes with time but retains its attention to clarity and detail in a certain fundamental classical field. There is hardly any statics, vectors are used where convenient, Lagrange's equations appear early (p. 72) and are used extensively, Hamilton's equations and Cartan's integral invariants are discussed, three chapters are devoted to the mechanics of continua, and there is a brief chapter (16 pp.) on relativity which leads the reader to the Schwarzschild field and the rotation of perihelion. The plan of the book is indicated by the chapter headings of Part I: I. Theory of vectors. II. Kinematics. III. Geometry of masses and kinetics. IV. Principles and general theorems of dynamics. Applications to the solid body. V. Elements of analytical dynamics. VI. Usual cases of integration. Stability. Small movements. Principle of least action and applications. VII. Shocks and percussions. VIII. Problems of dynamics without friction. IX. Supplementary material on analytical dynamics. X. Mechanics of strings. XI. Deformable media. XII. Mechanics of fluids. XIII. Velocity of light, space-time, gravitation. Part II is devoted largely to a careful treatment of the classical theory of friction, including the collision of rough bodies. Throughout the book there are a great many particular problems, most of them worked out, the intention being to give readers (including those who work alone) a thorough familiarity with the field. There is an index, but it is not, in the opinion of the reviewer, sufficiently detailed, and reference to pages, rather than to paragraphs, would have made it easier to use.

J. L. Synge (Dublin).

Kolčin, N. I. Curvature of conjugate surfaces in spatial gearings. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **13**, no. 49, 5–15 (1953). (Russian)

The differential geometry of a pair of mating surfaces is illustrated by the example of a worm (whose thread section is the same as an involute gear tooth) driving a worm-wheel. The coefficients of the differential quadratic forms of the first and second orders are developed from the equations of the mating surfaces. The curvature of a normal section of a surface is then expressed in terms of these coefficients. In particular, for the contact point on the normal from the center of the wheel to the axis of the worm, the radius of curvature K of the wheel tooth, on the normal section parallel to the wheel axis, is $R \sin \delta$, where R is the pitch radius and δ is the pressure angle. *M. Goldberg.*

Rotinyan, L. A. Construction of the positions of spatial three-link mechanisms. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **14**, no. 53, 11–19 (1953). (Russian)

Čerenin, V. P. Symbolic representation and the synthesis of regular toothed mechanisms consisting only of members with fixed axes of rotation. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1954**, no. 3, 24–38 (1954). (Russian)

Several toothed circular gears are mounted on a fixed shaft and are meshed with another set mounted on another fixed shaft. Various transmission ratios are obtained by the various methods of connecting, through clutches, the gear pairs into a gear train. The systematic method of obtaining all the possible connections, and therefore all the possible transmission ratios, involves a study of all the connections similar to the possible graphs and trees. Tabulated schemes up to seven clutches are presented. An illustrated example using four gear pairs and five clutches gives six ratios; 26 arrangements give a set of ratios of the type a, b, c, pa, pb, pc, while 24 arrangements give a set of the type $1, a, b, p, pa, pb$.

M. Goldberg (Washington, D. C.).

Rumyancev, V. V. On stability of rotation of a heavy rigid body with one fixed point in S. V. Kovalevskaya's case. Akad. Nauk SSSR. Prikl. Mat. Meh. **18**, 457–458 (1954). (Russian)

The Kowalevski top ($A=B=2C$ with equatorial mass center) can spin uniformly about the vertical through the fixed point O if the mass center C remains on it. Lyapunov's "second method" is used to show that this motion is stable if, and only if, C is below O. *A. W. Wundheiler.*

Masotti, Arnaldo. Su alcune questioni di media nei moti centrali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **15** (1953), 381–386 (1954).

This paper deals with the general properties of motion under the influence of a central force. Definitions for true and mean anomaly and the equation of the center, well-known in elliptic motion, are given for this more general case. They are denoted by V, M and C, respectively. If G is a function of the position only, the relationship is given between the mean G_M of this function with respect to M and the mean G_V with respect to V. Using for G specific functions general properties of the motion are derived.

A. J. J. van Woerkom (New Haven, Conn.).

Hydrodynamics, Aerodynamics, Acoustics

Braun, I. The momentum equations of the Reiner liquid. Rend. Circ. Mat. Palermo (2) **2** (1953), 258–265 (1954).

The author substitutes the constitutive equations of a Reiner-Rivlin fluid in Cauchy's equations of motion, specializing the result to physical components in orthogonal co-ordinates. *C. Truesdell* (Bloomington, Ind.).

Whitham, G. B. A note on a paper by G. C. McVittie. Quart. Appl. Math. **12**, 316–318 (1954).

The author gives a simple derivation of McVittie's [same Quart. **11**, 327–336 (1953); these Rev. **15**, 175] solution of the equations of conservation of mass and momentum for a perfect fluid. The derivation does not involve the connection between these solutions and Einstein's field equations in general relativity. *A. H. Taub* (Urbana, Ill.).

***Vladimirsky, Serge. Sur le potentiel des vitesses généralisé et son application en hydrodynamique.** Mémoires sur la mécanique des fluides offerts à M. Dimitri P. Riabouchinsky, pp. 413–416. Publ. Sci. Tech. Ministère de l'Air, Paris, 1954. 3000 francs.

The author exhibits a stream function ψ and a corresponding Riabouchinsky's generalized velocity potential [C. R. Acad. Sci. Paris **222**, 426–428 (1946); these Rev. **7**, 354] for which the vector field $u=\partial\psi/\partial y$, $v=-\partial\psi/\partial x$ tends to uniformity at infinity and to $u=v=0$ on a circle. Neither this paper nor the reference mention any special virtues of the generalized potential. The work is of doubtful value, for since only the equation of continuity for incompressible plane flow has been satisfied, very artificial body forces would have to be introduced to satisfy the remaining hydrodynamic equations. *J. H. Giese* (Ann Arbor, Mich.).

Levin, E. Note on the circle theorem of hydrodynamics. Quart. Appl. Math. **12**, 315–316 (1954).

The author generalizes the reviewer's circle theorem [Theoretical hydrodynamics, 2nd ed., Macmillan, New York, 1950, p. 149; these Rev. **11**, 471] to include a rigid boundary whose inverse with respect to the circle is a streamline of the undisturbed flow. *L. M. Milne-Thomson.*

Morgan, G. W. On the non-steady motions of a rigid body in an ideal fluid. Quart. Appl. Math. **12**, 277–285 (1954).

Does there exist a point of a solid body such that the body, being acted upon by a force which has a constant direction and whose line of action passes through the point, will move in an inviscid liquid with pure translation parallel to the force? The author uses Kirchhoff's equations to show that such a point exists only when certain symmetry conditions are satisfied. *L. M. Milne-Thomson* (Greenwich).

***Tomotika, Susumu, and Hasimoto, Zirô. Studies of discontinuous flow. I. Discontinuous flow past a flat plate near a plane wall.** Sûributurigaku kenkyu. 1. Ryutairikigaku no syomondai. I. [Investigations in mathematical physics. Vol. 1. Problems of hydrodynamics, I.] Pp. 1–21. Iwanami shoten, Tokyo, 1950. 480 yen. (Japanese)

The authors consider a two-dimensional discontinuous flow of Helmholz-Kirchhoff type past a flat plate placed near a boundary wall in order to investigate the influence of the wall upon the drag of the plate. The drag as well as several quantities concerning the configuration of the plate are determined in terms of elliptic theta-functions. The change of the drag is discussed by numerical calculation for some values of the attack angle and of the relative length of the plate. *Y. Komatu* (Tokyo).

***Tomotika, Susumu, and Hasimoto, Zirô. Studies of discontinuous flow. II. Bobyleff's problem.** Sûributurigaku kenkyu 1. Ryutairikigaku no syomondai I. [Investigations in mathematical physics. Vol. 1. Problems of hydrodynamics. I.] Pp. 22–42. Iwanami shoten, Tokyo, 1950. 480 yen. (Japanese)

It is shown that Bobyleff's result on a discontinuous flow past a bent plate placed symmetrically with respect to the principal stream-line can be re-obtained as a limiting case of the authors' result in the paper reviewed above, where, after an inversion with respect to the wall (the line of symmetry), the plate approaches the wall. *Y. Komatu.*

***Tomotika, Susumu, and Hasimoto, Zirô. Studies of discontinuous flow. III. Discontinuous flow past a flat plate which touches a plane wall with its trailing edge.** Sûributurigaku kenkyu. 1. Ryutairikigaku no syomondai. I. [Investigations in mathematical physics. Vol. 1. Problems of hydrodynamics. I.] Pp. 43–55. Iwanami shoten, Tokyo, 1950. 480 yen. (Japanese)

For a flow of the same type as considered by Bobyleff, expressions are established for the drag and the moment. By means of numerical calculation it is concluded that the drag increases with the width of the dead water. *Y. Komatu.*

***Endô, Dyûrô. Problems of two flat plates in hydrodynamics. I. On the forces exerted upon two flat plates in a uniform flow.** Sûributurigaku kenkyu. 2. Ryutairikigaku no syomondai. II. [Investigations in mathematical physics. Vol. 2. Problems of hydrodynamics. II.] Pp. 1–23. Iwanami shoten, Tokyo, 1952. 650 yen. (Japanese)

Let two plates be laid in a two-dimensional uniform flow of a perfect fluid, their position being arbitrary. Following Garrick's method of conformal mapping, the author derives explicit formulas for the forces exerted upon the plates by means of Blasius' general formula. Several formulas, mainly those concerning elliptic functions, necessary in the present and subsequent papers are listed at the end of the paper.

Y. Komatu (Tokyo).

***Endô, Dyûrô. Problems of two flat plates in hydrodynamics. II. On the moments of pressure exerted upon two flat plates in a uniform flow.** Sûributurigaku kenkyu. 2. Ryutairikigaku no syomondai. II. [Investigations in mathematical physics. Vol. 2. Problems of hydrodynamics. II.] Pp. 25–36. Iwanami shoten, Tokyo, 1952. 650 yen. (Japanese)

Continuation of the paper reviewed above. The author derives explicit formulas for the moments of pressure exerted upon the plates by means of Blasius' second general formula. In a particular case when two plates are parallel, an alternative method is given by means of Laurent series expansion. *Y. Komatu* (Tokyo).

***Endô, Dyûrô. Problems of two flat plates in hydrodynamics. III. On the actions exerted upon two flat plates in a uniform flow.** Sûributurigaku kenkyu. 2. Ryutairikigaku no syomondai. II. [Investigations in mathematical physics. Vol. 2. Problems of hydrodynamics. II.] Pp. 37–65. Iwanami shoten, Tokyo, 1952. 650 yen. (Japanese)

Using analytical expressions which have been derived in the two previous papers reviewed above, the author performs numerical computations. Several dynamical coefficients are determined and then discussed by means of numerical results. *Y. Komatu* (Tokyo).

***Tasiro, Yosirô. On the hodograph solution of the flow with free stream-lines.** Sûributurigaku kenkyu. 2. Ryutairikigaku no syomondai. II. [Investigations in mathematical physics. Vol. 2. Problems of hydrodynamics. II.] Pp. 189–219. Iwanami shoten, Tokyo, 1952. 650 yen. (Japanese)

The author attacks the problem of two-dimensional irrotational flow of an incompressible perfect fluid for which the boundary consists of a piecewise straight wall and free stream-lines. It is reduced to determining suitably the distribution of sources, sinks, doublets or, in general, multiple sources, in the hodograph plane. Several types of the problem are illustrated in detail. *Y. Komatu* (Tokyo).

Eppler, R. Beiträge zu Theorie und Anwendung der unstetigen Strömungen. J. Rational Mech. Anal. **3**, 591–644 (1954).

The author is concerned with the determination of plane cavity flows, especially in the case of an infinite stream past a barrier. The inverse problem, that is, the determination of the class of all such flows, is solved by putting them in correspondence with all logarithmic hodograph-diagrams satisfying certain conditions. According to the author this has the advantage over other solutions proposed that many qualitative features of specific flows may at once be seen without further calculations. The distribution of inflection points on the fixed and free boundaries, as well as other qualitative properties, are considered in some detail. For the determination of cavity flows past a given barrier (direct problem) the author suggests the iteration process of C. Schmieden [Ing.-Arch. **3**, 356–370 (1932); **5**, 373–375 (1934)], and makes some practical considerations for its use. No attempt is made to prove convergence. A couple of examples with rectilinear and circular profiles are worked out. Finally, to describe viscous wakes with high Reynolds numbers, the author introduces a generalization of the Helmholtz flow having a cavity of finite width confined between two parallel lines and cavitation number -1. This gives the drag the value Mv_∞, where M is the mass flux across any transversal line, as approximately derived from the physical assumptions. A comparison with experimental facts is given at the end. *E. H. Zarantonello.*

Roshko, Anatol. A new hodograph for free-streamline theory. NACA Tech. Note no. **3168**, 39 pp. (1954).

In order to account theoretically for the fact that the observed values of the drag of bluff bodies are larger than those predicted by the classical Helmholtz wake theory, the author postulates a model of the wake in which, unlike the classical theory, the flow speed on the free streamline exceeds the speed at infinity. In this model the free streamlines extend a finite distance downstream where they are joined to semi-infinite straight line portions of the wake boundary. The hodograph image of the resulting flow is the "new hodograph" of the title. The author carries out by the usual methods of conformal mapping explicit calculations for the flat plate and symmetric wedge, and numerical calculations for the circular cylinder, also making certain comparisons with experiment.

In common with the Riabouchinsky and reentrant jet models, which achieve the same ends but have the advantage of being finite in extent, the model described here requires that the flow speed on the free boundary be assigned beforehand. From the physical viewpoint such an assignment is entirely arbitrary in the theory of wakes, but is quite natural in cavitation problems. *D. Gilbarg* (Stanford, Calif.).

Marchi, Enrico. Sui fenomeni di efflusso piano da luci a battente. Ann. Mat. Pura Appl. (4) **35**, 327–341 (1953).

The author gives an approximate treatment of the symmetric jet efflux from a polygonal channel taking gravity into account. His approximation consists in replacement of the unknown image of the free boundary in the hodograph plane by a known curve so that explicit conformal mapping procedures can be applied. *D. Gilbarg.*

Sretenskiĭ, L. N. The motion of a vibrator under the surface of a fluid. Trudy Moskov. Mat. Obšč. **3**, 3–14 (1954). (Russian)

Let a source of strength $Q \cos \sigma t$ move rectilinearly with constant velocity c at a distance h beneath the undisturbed surface of a heavy fluid. Using the method of stationary phase, the author investigates the asymptotic form of the free surface at a great distance from the source. The problem separates into two cases: 1) $g > 4\sigma c$; 2) $g < 4\sigma c$. The result for 1) is somewhat similar to the case treated by Kelvin when the source strength is constant. The pattern for case 2) is more complex and involves an extra wedge-shaped region containing longitudinal waves. *J. V. Wehausen.*

Marnyanskiĭ, I. A. Diffraction of waves about a submerged vertical plate. Akad. Nauk SSSR. Prikl. Mat. Meh. **18**, 233–238 (1954). (Russian)

The plate of the title occupies the portion of the y-axis between $-\infty$ and $-T$. The problem is treated two-dimensionally and the surface waves are assumed infinitesimal. This leads to a boundary-value problem for the velocity potential. The same problem has been treated before by Dean [Proc. Cambridge Philos. Soc. **41**, 231–238 (1945); these Rev. **8**, 110] and Ursell [ibid. **43**, 374–382 (1947); these Rev. **9**, 117] of whose work the author was apparently unaware. He arrives at the same result for the transmission coefficient and phase shift. New is a treatment of the surface profile near the barrier. *J. V. Wehausen.*

Kostyukov, A. A. On formulas for computation of the wave resistance and lifting force of bodies submerged in a liquid. Akad. Nauk SSSR. Prikl. Mat. Meh. **18**, 225–232 (1954). (Russian)

Let a body S be submerged beneath a free surface in a steady stream of fluid. The velocity potential may be written

$$\varphi(x, y, z) = -\frac{1}{4\pi}\int_S \frac{q(\xi, \eta, \zeta)}{r} dS + F(x, y, z),$$

where r is the distance from (x, y, z) to (ξ, η, ζ). Then, from Lagally's theorem one can write the force vector on the body in the form

$$P = -\rho \int_S q \operatorname{grad} F dS.$$

Starting with known formulas for φ for a submerged source, the author uses this equation to derive formulas for the wave resistance, vertical force and pitching moment for submerged bodies with a vertical plane of symmetry. They are derived for the case of water of finite depth.

J. V. Wehausen (Providence, R. I.).

Khare, R. C. A case of tidal waves on an open sheet of water of variable depth. Proc. Nat. Acad. Sci. India. Sect. A. **21**, 81–97 (1952).

Using the linearized shallow-water theory, the author treats the wave motion in a basin with vertical walls forming a circular cylinder and a corrugated bottom, $h = h_0[1 + m \sin(r\pi/2a)]$. It is assumed that the free surface may be written $f(r) \cos s\theta$ and that m is small. A solution is obtained by a method of successive approximation in which successively higher powers of m are retained. In a final section an angular velocity is imposed upon the fluid. The approximate solutions are expressed as power series and are not very perspicuous. *J. V. Wehausen.*

Frank, Wilhelm. Zur Berechnung von Potentialströmungsfeldern. Österreich. Ing.-Arch. **8**, 97–107 (1954).

A method of computing the inviscid, potential, incompressible flow about a closed two-dimensional form is proposed. An arbitrary closed profile within the region interior to the given form is assumed, and distributions of sources and sinks and vorticity on this profile are to be determined so as to satisfy Laplace's equation exterior to the given form, and the boundary conditions. Integral equations of the first kind are derived for determining each of these distributions. It is also shown that, as the internal profile approaches the given form, these equations become the well-known integral equations of the second kind satisfied by surface distributions on a boundary. It is proposed to solve each of these integral equations by transforming it, by means of a Fourier analysis, into an infinite set of linear equations in an infinite number of unknowns, but solving only a finite array of these equations and unknowns in order to obtain an approximate solution.

Questions of the convergence of the approximate solutions for larger arrays of equations, and the existence of the solutions of the integral equations of the first kind are briefly discussed. It is clear from other considerations, however, that there will be cases where the solutions do not exist and hence the approximations will not converge. This will certainly be the case when the analytical continuation of the flow into the body encounters an isolated singularity outside of the assumed inner profile.

The method is applied to discuss the flow about a circular cylinder enlarged by the amount of its displacement thickness. *L. Landweber* (Iowa City, Ia.).

Neumark, S. Velocity distribution on thin bodies of revolution at zero incidence in incompressible flow. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2814** (1950), 42 pp. (1954).

To approximate the flow about the body $r^2 = F(x)$, $|x| \leq b$, the author superposes uniform flow at velocity $-U$ parallel to the x-axis and the flow from a distribution of sources of intensity $q(x) = -\pi U dF/dx$ on the x-axis. The velocity components are expressed in the form

$$4v_r/U = -F'(x)rI_0 - rKI_1, \quad 4v_x/U = \sum_1^\infty (d^kF/dx^k)I_k/(k-1)!$$

for some $K(x)$ and $I_k(x, r) = \int_{-b-x}^{b-x} \rho^k (\rho^2 + r^2)^{-3/2} d\rho$. If terms of order r^2 and $r^2 \log r$ are neglected,

$$r^2 I_0 = 2, \quad I_1 = -2x/(b^2 - x^2), \quad I_2 = -2 + \log[4(b^2 - x^2)/r^2],$$
$$kI_{k+2} = (b-x)^k + (-b-x)^k,$$
$$4v_r/U = -F'rI_0, \text{ and } \Delta V/U = (V-U)/U = -v_x/U + v_r^2/2U^2,$$

where V is the local speed of flow. If d^kF/dx^k is considered to be of the order of ϑ^2 where ϑ is the body's thickness ratio, the foregoing approximations yield $\Delta V/U$ on the body with an error of $\vartheta^4 \log \vartheta$, ϑ^4, or higher. The author chooses polynomials of degree ≤ 8 for F and presents extensive computations of $\Delta V/U$ on several families of bodies $r^2 = F = \vartheta^2 \sum_0^6 A_n x^n$ to show the importance of the v_r^2 term and effects of ϑ, and nose and tail shape. *J. H. Giese* (Ann Arbor, Mich.).

Hancock, G. J. The design of thin finite wings in incompressible flow. Aeronaut. Quart. **5**, 119–130 (1954).

The shape of the mean-camber surface of a wing for predetermined load distribution is calculated, within the scope of the usual linearized theory for incompressible flow, by standard methods. Three numerical examples are worked out, in each of which it is assumed that the lift loading drops to zero at the leading edge. *W. R. Sears.*

Spence, D. A. Prediction of the characteristics of two-dimensional airfoils. J. Aeronaut. Sci. **21**, 577–587, 620 (1954).

Basically, the procedure used is to calculate the potential flow about a body representing the airfoil plus its boundary layer and viscous wake. This is the same aim as Preston's [Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda nos. **1996** (1943); **2107** (1945); **2725** (1953); these Rev. **9**, 115], but the analysis is somewhat different. The displacement effects of boundary layer and wake are represented by source distributions. The circulation is determined by the following condition: Assuming the boundary-layer thickness and profile to be known, the pressures at two points located above and below the trailing edge along normals to the ideal (Kutta-Joukowski) flow are estimated. These two points are selected at the outer edges of the upper- and lower-surface boundary layers; thus the velocities there must be given by Bernoulli's equation. This condition therefore relates the velocities at the two points and the circulation must have the value that satisfies this relation. This estimate has been checked by experiments. Results for a Joukowski profile at three Reynolds numbers are given. The difficulty of predicting transition-point location is mentioned, and brief paragraphs are included regarding grossly separated flow patterns, uniqueness of solution, and the effects of compressibility. *W. R. Sears.*

Hawthorne, W. R. The secondary flow about struts and airfoils. J. Aeronaut. Sci. **21**, 588–608, 648 (1954).

A cylindrical strut is placed in an incompressible frictionless flow; the axis of the cylinder is parallel to the z-axis and the flow far upstream ($x=-\infty$) is a parallel shear flow in the x-direction described by the velocity $U(z)$. The flow pattern at every z is supposed to be a small perturbation of the plane flow at stream speed $U(z)$ about the same cylinder. The components of the vorticity perturbation are calculated in terms of the properties of the basic plane flow and the function $U'(z)$. An important result is the appearance of a streamwise component of vorticity in the flow far downstream, i.e., "secondary flow." (This is reminiscent of the same author's earlier work on secondary flow in curved passages [Proc. Roy. Soc. London. Ser. A. **206**, 374–387 (1951); these Rev. **13**, 177].) Application of the theory is made to bicusped, biconvex (lenticular), and elliptical profiles, and a thin circular-arc airfoil producing lift. Secondary-flow effects are much reduced by avoiding blunt leading edges. The results of some experiments are presented; these confirm the predictions of the theory, in general. It is pointed out that the drag attributable to secondary flow, in these experiments, was appreciably larger than the energy in the secondary flow. *W. R. Sears* (Ithaca, N. Y.).

Castoldi, Luigi. Sui moti di fluidi reali per cui si verifica una esatta linearizzazione della equazione dinamica. Atti Accad. Ligure **9** (1952), 222–227 (1953).

The author calls attention to the well known fact that the acceleration field reduces to the local time derivative if and only if the stream-lines are straight and the speed is instantaneously constant along them, implying that the surfaces of constant speed contain both the stream-lines and the vortex-lines. He finds that Navier's equations possess a certain pseudo-plane solution of this kind.
C. Truesdell (Bloomington, Ind.).

Levey, H. C. Two dimensional source flow of a viscous fluid. Quart. Appl. Math. **12**, 25–48 (1954).

The author studies the behavior of a steady two-dimensional source flow of a viscous heat-conducting gas as the viscosity and thermal conductivity approach zero. He finds that in the limit all physically significant solutions contain ideal shocks joining parts of inviscid solutions, and, unlike the ideal source flow, do not contain limit lines. Another, and somewhat surprising, conclusion is that within the shock the entropy has a maximum which does not disappear as the viscosity approaches zero. In the details the author confines his attention principally to the case of constant viscosity and thermal conductivity and to a value of the Prandtl number (which is approximately $\frac{3}{4}$ at small viscosity) for which the equations of motion have an explicit first integral. The resulting single first-order equation describing the possible flows is then studied by means of its direction field. The details are quite different from those for the well known theory of the one-dimensional shock profile.
D. Gilbarg (Stanford, Calif.).

Pieruschka, Erich. Die mathematischen Grundlagen zu einer Messmethode des Schubmoduls zäher Flüssigkeiten. Z. Angew. Math. Mech. **31**, 83–92 (1951). (English, French and Russian summaries)

In a visco-elastic theory of Maxwell type as formulated by Fromm [same Z. **28**, 43–54 (1948); these Rev. **10**, 750], the author gets a linearized solution representing the transmission of forced harmonic oscillation of a plate through the medium to a parallel free plate. He attempts to evaluate the error arising from taking the amplitude as small. He applies the results to experiments in which the parallel plates are replaced by coaxial cylinders. The object was by measurement of normal stresses to determine whether the classical ("Newtonian") theory of fluids, a limit case of the visco-elastic fluid, satisfactorily describes all fluids tested. The author concludes that for five fluids, including oil and tar, it does not, while for most the results are inconclusive. [The reviewer prefers to interpret the experimental results without bias in favor of any particular theory, especially not linearized visco-elasticity, as simply showing the presence in ordinary fluids under ordinary circumstances of a secondary effect typical of all generalized or non-linear theories.] *C. Truesdell* (Bloomington, Ind.).

Hasimoto, Hidenori. Rayleigh's problem for a cylinder of arbitrary shape. J. Phys. Soc. Japan **9**, 611–619 (1954).

This paper deals with the motion of an incompressible viscous fluid generated by an infinite cylinder moving impulsively from rest. For arbitrary section, the problem is reduced to that of circular cylinder by conformal mapping. Asymptotic expressions, such as stream function in the vicinity of the cylinder, and drag are derived. A paper by Batchelor treats the same subject but by a different method [Quart. J. Mech. Appl. Math. **7**, 179–192 (1954); these Rev. **16**, 83]. *Y. H. Kuo* (Ithaca, N. Y.).

Cocchi, Giovanni. Il moto laminare vario in tubi cilindrici di sezione circolare. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **15** (1953), 393–401 (1954).

The paper gives a full discussion of the flow, uniform along the tube but not constant in time, of a viscous incompressible fluid in a tube of uniform circular cross-section. Exponentially damped solutions involving Bessel functions are obtained for the case when no pressure gradient is acting. Solutions are also obtained for various time-dependent forms of external force. The results are compared with those obtained by assuming an always parabolic distribution of velocity across the tube; this necessarily underestimates the rate of energy dissipation. *M. J. Lighthill.*

Wundt, Hermann. Abklingen eines schwachen Dralls in der Längsströmung zwischen koaxialen Kreiszylindern. Z. Angew. Math. Physik **5**, 270–276 (1954).

Collatz and Görtler [same Z. **5**, 95–110 (1954); these Rev. **16**, 83] examined the decay of a small axially symmetrical swirl superimposed upon steady laminar flow at high Reynolds number in a straight circular tube. The method is here extended to flow between coaxial circular cylinders. It is found that the damping of the angular momentum increases rapidly with increasing radius of the inner cylinder.
D. C. Pack (Glasgow).

Scheidegger, Adrian E. Statistical hydrodynamics in porous media. J. Appl. Phys. **25**, 994–1001 (1954).

This lengthy discussion of the physical bases of Darcy's law and allied topics adds little to one's understanding of flow through porous media. Great emphasis is put on the fact that the results are ensemble averages over a large number of different but macroscopically identical specimens of porous material, but the neglect of any autocorrelation between successive movements of particles of fluid detracts greatly from the interest of the statistical arguments. It is pointed out that the porous medium introduces a "disper-

sion" (or, more properly, a diffusion) of individual particles of fluid across the mean streamlines, though of course this does not affect the equations of the mean motion.

M. J. Lighthill (Manchester).

Crabtree, L. F. The compressible laminar boundary layer on a yawed infinite wing. Aeronaut. Quart. **5**, 85–100 (1954).

This is an approximate calculation of the effects of compressibility on the laminar boundary layer on a yawed cylinder. The "independence principle" that simplifies the corresponding incompressible problem does not apply. The familiar Stewartson-Illingworth transformation, which assumes insulated surface and Prandtl number equal to one, is applied to the boundary-layer differential equations. This puts the equation for the cross-flow components $u(x, z)$ and $w(x, z)$ into a form resembling the incompressible case but with an extra term involving the axial component $v(x, z)$ (y is the axial coordinate). The equation for v, after transformation, involves u and w and is the same as for incompressible flow. The extra term mentioned depends on the Mach number of the axial component of potential flow and is small when this component Mach number is less than one. The author therefore proposes a successive-approximation procedure in which this term is neglected in the first step. The method is applied to "stagnation-point flow" wherein $U(x)$, the cross-flow potential velocity is proportional to x. For one value of the axial-Mach-number parameter, the first approximation to $v(x, z)$ is found and is used to find the first-order correction to $u(x, z)$. The second-order approximation to $v(x, z)$ is then worked out. The general Falkner-Skan case $U(x)=cx^K$, c and K being constants, is considered next. The equations are set up but not solved; it is concluded that the same method of successive approximations is valid.

Proceeding to momentum methods, the author makes two alternate proposals. The first is to assume isothermal flow, for which the "independence principle" holds. It is shown, in a typical plane-flow case, that the isothermal assumption accounts for most of the effect of compressibility. The other approximation is the same as the first step in the differential-equation procedure previously discussed; it consists of using the Stewartson-Illingworth transformation and then neglecting the rest of the effects of compressibility on the cross-flow boundary layer. This approximation permits calculation of the three-dimensional flow pattern by means of very simple formulas. *W. R. Sears.*

Chow, Tse-Sun. On the average second moment of the energy spectral intensity. Quart. Appl. Math. **12**, 287–294 (1954).

Considering solutions of the Navier-Stokes equations (for an incompressible viscous fluid without boundaries) represented by trigonometric series, the author introduces the average second moment of the energy spectral density; for a two-dimensional flow the main result is that this moment is a decreasing function of time; a lower bound of the kinetic energy of the fluid (by unit volume) is given for two- and three-dimensional flows. *J. Kampé de Fériet.*

Ghosh, K. M. A note on the Kármán's spectrum function of isotropic turbulence. Proc. Nat. Inst. Sci. India **20**, 336–340 (1954).

W. Heisenberg a proposé une équation pour l'évolution de le fonction spectrale $F(k, t)$ de la turbulence isotrope, et il en a indiqué une solution satisfaisant à une loi de similitude. N. R. Sen a obtenu ensuite, pour les petites valeurs de ν, une solution plus générale de la forme

$$F(k, t)=\frac{1}{K^2}\frac{1}{k_0^3 t_0^2}\left(\frac{t_0}{t}\right)^{2-4\epsilon} f\left[\frac{k}{k_0}\left(\frac{t}{t_0}\right)^{\epsilon}\right].$$

L'auteur montre qu'une solution de ce type convient aussi à l'équation proposée par Karman, et qui contient comme cas particulier l'équation de Heisenberg. Seule la forme de le fonction f dépend des paramètres contenus dans l'équation de Karman. L'auteur étudie ensuite le comportement asymptotique de $f(x)$ pour les petites et les grandes valeurs de f. *J. Bass* (Chaville).

Birkhoff, Garrett. Fourier synthesis of homogeneous turbulence. Comm. Pure Appl. Math. **7**, 19–44 (1954).

Les deux premières parties de ce travail sont consacrées à une discussion algébrique de la structure d'un champ de vitesses aléatoires. L'auteur montre qu'il existe une mesure gaussienne et une seule qui admette les mêmes moments d'ordre 1 et 2 en n points qu'un champ défini par une mesure de probabilité donnée. Ce résultat valable dans un espace vectoriel à $3n$ dimensions, s'étend à l'espace de Hilbert. On peut toujours trouver une base f_i telle que l'élément aléatoire f soit de la forme $\sum c_i f_i$, les c_i étant indépendantes, normales, avec $\bar{c}_i=0$, $\sum \bar{c}_i^2<\infty$.

Dans la troisième partie, les résultats théoriques sont traduits en langage mécanique. Ils signifient que, pour une distribution d'énergie $\delta(k)$ donnée, il existe une distribution gaussienne de vitesse compatible avec la turbulence homogène, isotrope, incompressible, telle que les quantités dépendant linéairement de la vitesse soient normalement distribuées, et que les différentes composantes spectrales soient indépendantes. Les conséquences de ce résultat ne sont que partiellement vérifiées par l'expérience. Par exemple, la statistique normale est réversible (le changement du signe de la vitesse en chaque point conserve la mesure), et ceci impliquerait la disparition des corrélations triples dans l'équation classique de Kármán-Howarth. En admettant un "principe d'approximation," d'après lequel, dans un domaine fermé, une statistique normale est limite de statistiques à spectre discret, on peut établir l'identité entre les moyennes spatiales et les moyennes d'ensemble.

La quatrième partie est une étude plus poussée de l'état final de la turbulence, auquel la structure gaussienne est susceptible de s'appliquer. L'auteur prouve que, si $E(k)$ est la densité d'énergie, $E(0)$ est indépendant du temps (sous certaines hypothèses de convergence). La loi $E(k)=O(k^n)$ ne lui semble pas nécessaire, et il montre qu'elle n'a pas tendance à s'établir spontanément au cours du temps. L'invariant de Loitsyansky serait une intégrale divergente. Cette partie se termine par une discussion des résultats experimentaux. *J. Bass* (Chaville).

Frankl', F. I. On a class of solutions of the gas dynamic equations of S. A. Čaplygin. Moskov. Gos. Univ. Učenye Zapiski **154**, Mehanika **4**, 287–310 (1951). (Russian)

The author constructs solutions of the Chaplygin hodograph equation which represent, when transferred to the physical plane, a potential flow past a blunt symmetric profile, with Mach number 1 at infinity. Actually only the upstream part of the flow is computed, which is bounded by the nose of the profile and two Mach lines ("limiting characteristics") which meet the sonic line at infinity. The solution stream-function ψ consists of a principal term $\hat{\psi}$ and

correction terms. The principal term determines the character of the singularity corresponding to the point at infinity. It is of the form

$$\hat{\psi}=\rho^{\frac{1}{3}}f(\theta/\rho),\quad \rho^2=\theta^2+4\eta^3/9$$

and must satisfy the Tricomi equation $\eta\hat{\psi}_{\theta\theta}+\hat{\psi}_{\eta\eta}=0$. Here θ is the inclination of the velocity vector and η a fixed function of the speed. In order that $\hat{\psi}$ be regular along the limiting characteristics one must have $k=\frac{1}{3}+2m$, $m=0, \pm 1, \cdots$. The requirement that all streamlines intersect the sonic line leads to the values $m=-1$, $k=-5/3$. The function f may then be chosen as an algebraic function:

$$f(s)=(1-s)^{1/3}[\tfrac{1}{3}+s]-(1+s)^{1/3}[\tfrac{1}{3}-s].$$

For the principal part of the velocity potential one obtains the expression $\hat{\phi}=\rho^{-4/3}g(\theta/\rho)$. The author concludes that the disturbance velocities die out like $|y|^{-2/5}$ along the limiting characteristics and along the sonic line and like $|x|^{-3/4}$ in front of the profile.

The correction terms are chosen so that ψ satisfies the Chaplygin equation. The author shows that there exists a rigorous solution which admits, in the subsonic range, the expansion

$$\psi=\sum_{n=2}^{\infty}(\beta_0 n^{2/3}+\beta_1+\beta_2 n^{-2/3})\frac{z_{n/2}(\tau)}{z_{n/2}(\tau^*)}\cos n\theta.$$

Here τ is the Chaplygin variable ($\tau=q^2/q^2_{\max}$, $\tau=\tau^*$ for $M=1$) and z, the Chaplygin function.

These results were announced in Doklady Akad. Nauk SSSR (N.S.) **57**, 661–664 (1947) [these Rev. **9**, 478] and the author remarks that this establishes his priority over Guderley [cf. G. Guderley and H. Yoshihara, J. Aeronaut. Sci. **17**, 723–735 (1950); these Rev. **15**, 264]. *L. Bers.*

Ludford, G. S. S., and Martin, M. H. One-dimensional anisentropic flows. Comm. Pure Appl. Math. **7**, 45–63 (1954).

The authors prove the following theorems. (1) For any gas the local speed of sound is the geometric mean of the rates at which pressure and velocity are propagated with respect to the gas. For a polytropic gas the rate at which the absolute temperature is propagated with respect to the gas is a weighted harmonic mean of the rates at which density and pressure are propagated with respect to the gas. (2) The Monge-Ampère equation $\xi_{\psi\psi}\xi_{pp}-\xi^2_{\psi p}+\lambda^2=0$, $\lambda=\Psi(\psi)P(p)$, has intermediate integrals only in the following cases: $\lambda=0$, $\lambda=1$, $\lambda=P(p)\neq 0$, $\lambda=p^{-2}$, $\lambda=\psi^{m-1}/p^{m+1}$ ($m\neq 0$), $\lambda=\psi^{-1}/p$, $\lambda=e^{\psi}e^{p}$ and in those which arise from these under translations, reflections in the line $p=\psi$, and dilatations in the ψ and p directions, in the (ψ, p)-plane. (3) The only anisentropic simple waves in a polytropic gas ($\lambda\neq 1/3, \pm 1$) arise from the solutions of the intermediate integrals

$$(*)\quad u\psi+tp-\xi-\frac{k}{m}\left(\frac{\psi}{p}\right)^m=r,\quad u\psi+tp-\xi+\frac{k}{m}\left(\frac{\psi}{p}\right)^m=s,$$

$$m=\tfrac{1}{2}(\gamma^{-1}-1),\quad S-S_0=c_p\log\left\{\frac{k^2\psi^{2m-2}}{2m+1}\right\}.$$

They require the entropy distribution (*), so that

$$\tau=\left(\frac{k^2}{2m+1}\right)\frac{\psi^{2m-2}}{p^{2m+1}}.$$

The unsteady one-dimensional anisentropic, adiabatic flow of a gas with arbitrary equation of state is represented parametrically in the (x, t)-plane by

$$(**)\quad x=\int[(\xi_\psi\xi_{\psi p}+\tau)d\psi+\xi_\psi\xi_{pp}dp],\quad t=\xi_p,\quad u=\xi_\psi,$$

where $\xi=\xi(\psi, p)$ is a solution of the Monge-Ampère equation $\xi_{\psi\psi}\xi_{pp}-\xi^2_{\psi p}=\tau_p$ ($\tau=\tau(\psi, p)$=specific volume, p=pressure, u=velocity, ψ=stream function, S=entropy). Isentropic flows are characterized by S=const., or $\tau=\tau(p)$. For isentropic flows the intermediate integrals of the Monge-Ampère equation give rise to simple waves. This provides a natural extension of the idea of a simple wave to anisotropic waves: the flow presented by (**) when ξ is a solution of an intermediate integral of the Monge-Ampère equation is called a simple wave. The theorem (2) is restricted to polytropic gases, where

$$\Psi(\psi)=n^{\frac{1}{2}}\exp\{(S-S_0)/2c_p\},\quad P(p)=p^{-(n+1)/2}.$$

Whenever two intermediate integrals of the Monge-Ampère equation exist, its integration can be effected by Ampère's method. Applied to isentropic flows it yields classical results from a different viewpoint; applied to anisentropic flows it yields new results, and in particular a concrete example of anisentropic flow. Finally an appendix is given: Remarks on the Eurler-Poisson equation. *M. Pinl* (Cologne).

Serrin, J. B. Comparison theorems for subsonic flows. J. Math. Physics **33**, 27–45 (1954).

This paper extends to plane and axially symmetric subsonic flows certain comparison theorems and their consequences previously established by the author for incompressible fluids [Amer. J. Math. **74**, 492–506 (1952); J. Rational Mech. Anal. **1**, 563–572 (1952); **2**, 563–575 (1953); these Rev. **13**, 877; **14**, 325, 1137]. These results, combined with those of the reviewer on comparison methods in subsonic flows [ibid. **2**, 233–251 (1953); these Rev. **14**, 920], suffice (except for omissions noted by the author) to extend to subsonic flows most of the earlier results by the author [see above references] and reviewer [ibid. **1**, 309–320 (1952); these Rev. **13**, 877] in the qualitative theory of incompressible flows with free boundaries. The paper also contains a useful sharpening of the comparison theorems, so that, in particular, the theorems now apply equally well to points situated on or off the axis in axially symmetric flow, whereas in the earlier proofs points on the axis were exceptional. *D. Gilbarg* (Stanford, Calif.).

Naylor, Derek. The simple wave in rotational gas flow. J. Rational Mech. Anal. **3**, 415–433 (1954).

The author considers the problem of determining rotational iso-energetic plane flows with one-dimensional hodographs. These flows are defined by the condition that the velocity vector is a function of one independent variable. First, the author introduces the Crocco stream function, ψ, and shows that: (1) the vorticity is the product of a function of ψ, $G(\psi)$, and a power of the sound speed; (2) the entropy is the integral of $G(\psi)$. By forming the Legendre transform of ψ and requiring that the hodograph be one-dimensional, it is shown that there must exist a one-parameter family of straight lines, λ=constant, in the flow along each of which lines the velocity vector and sound speed are constant. M. H. Martin [J. Math. Physics **29**, 76–89 (1950); these Rev. **12**, 215] has shown that the entropy, S, must be of one of two special forms for rotational iso-energetic flows with simple waves. By use of the present theory, the author verifies this result and proves that $G(\psi)$ is either A or A/ψ

where A is a constant. The first case is shown to imply the result that the λ=constant lines are concurrent; in the second case, the lines are parallel [M. H. Martin, ibid. **29**, 263–281 (1951); these Rev. **12**, 874]. A family of concurrent lines is determined for $\gamma=7/5$, $A=\gamma^{-1}$, and the resulting flow is discussed. Interesting details of the result include determination of the: (1) regions of the plane where the flow is subsonic and supersonic; (2) critical values of a flow parameter. Further, the general class of parallel lines with λ=constant is determined and the flow discussed. The author concludes by discussing: (1) the similarity between the above plane flows and axial symmetric rotational flows with conical symmetry; (2) the limit lines for plane flows.

N. Coburn (Ann Arbor, Mich.).

***Munakata, Ken-iti. Successive approximations in the hodograph method for treating compressible flow.** Sûributurigaku kenkyu. 1. Ryutairikigaku no syomondai. I. [Investigations in mathematical physics. Vol. 1. Problems of hydrodynamics. I.] Pp. 93–129. Iwanami shoten, Tokyo, 1950. 480 yen. (Japanese)

A two-dimensional irrotational flow of a compressible perfect fluid is considered under the adiabatic change. After transforming the equations of motion into the hodograph plane, they become linear. By supposing that dependent variables are expansible into power series with respect to the square of the Mach number, the author establishes a method of successive approximation for solving the equations thus obtained. The method is illustrated by applying it to some examples. A critical Mach number at which a shock wave begins to occur is determined in every case.

Y. Komatu (Tokyo).

***Hasimoto, Zirô. On the solution of some boundary-value problems of compressible fluid flow.** Sûributurigaku kenkyu. 2. Ryutairikigaku no syomondai. II. [Investigations in mathematical physics. Vol. 2. Problems of hydrodynamics. II.] Pp. 141–187. Iwanami shoten, Tokyo, 1952. 650 yen. (Japanese)

For a two-dimensional irrotational steady flow of a compressible perfect fluid obeying the adiabatic law, it is shown that the fundamental equation satisfied by the stream function in the hodograph plane has particular solutions involving the hypergeometric function. The zero-points and asymptotic expansions are investigated for hypergeometric functions whose auxiliary arguments depend on a non-negative integer. Using these analyses, he then deals with three particular cases of a discontinuous flow of a compressible fluid with two plane walls. *Y. Komatu.*

Miles, John W. Linearization of the equations of non-steady flow in a compressible fluid. J. Math. Physics **33**, 135–143 (1954).

C. C. Lin, E. Reissner, and H. S. Tsien [same J. **27**, 220–231 (1948); these Rev. **10**, 162] have determined conditions under which the equation for non-steady irrotational plane compressible flow can be linearized. The author extends their analysis to flows about flat three-dimensional bodies to find that linearization is possible if all of δ, $k\delta$, $M\delta$, and $kM\delta$ are small relative to unity and one of $|1-M|$, k, or (aspect ratio)$^{-2}$ is large relative to $\delta^{2/3}$. Here M is free-stream Mach number corresponding to velocity U, and if l is some significant dimension in the direction of flight, then δ is the ratio of the greater of the thickness of the body or the amplitude of transverse motion to l, and $k=\omega l/U$ for harmonic oscillations of angular frequency ω. *J. H. Giese.*

Popp, Simona. Corrections de compressibilité dans le problème de Helmholtz. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **4**, 555–562 (1952). (Romanian. Russian and French summaries)

The Janzen-Rayleigh method of expansion in powers of M^2, applied previously to subsonic flows past fixed bodies, is extended in this paper to the Helmholtz flow past a flat plate, which is the first free-boundary problem to be so treated. The author obtains explicitly the approximation to order M^2, using the complex variable approach to the Janzen-Rayleigh approximation. His treatment of the boundary conditions for the free-surface problem is the essential innovation in the method. *D. Gilbarg.*

***Tamada, Kô. Studies of two-dimensional flow of gas with special reference to the flow through various nozzles.** Sûributurigaku kenkyu. 1. Ryutairikigaku no syomondai. I. [Investigations in mathematical physics. Vol. 1. Problems of hydrodynamics. I.] Pp. 56–80. Iwanami shoten, Tokyo, 1950. 480 yen. (Japanese)

In order to investigate an irrotational steady flow of a compressible perfect fluid which contains subsonic as well as supersonic parts, the equation of motion is first approximated suitably. The approximate equation thus obtained may then be regarded as the exact one for an imaginary fluid with a correspondingly modified characteristic. The general discussion is illustrated by several examples accompanied by numerical tables and figures.

Y. Komatu (Tokyo).

***Tamada, Kô. Studies of two-dimensional flow of gas with special reference to the singularities of the solutions of a certain partial differential equation of mixed type.** Sûributurigaku kenkyu. 1. Ryutairikigaku no syomondai. I. [Investigations in mathematical physics. Vol. 1. Problems of hydrodynamics. I.] Pp. 81–92. Iwanami shoten, Tokyo, 1950. 480 yen. (Japanese)

Continuing his work on a flow containing subsonic as well as supersonic parts [cf. the paper reviewed above], the author starts with a fundamental linear equation obtained in the hodograph plane. Two kinds of particular solutions are found, of which one may be derived by superposing another. The character of singularity is studied. Illustrative examples of flow are constructed. *Y. Komatu* (Tokyo).

***Tamada, Kô. On the two-dimensional flow of a hypothetical gas which obeys the adiabatic law approximately, with special reference to Taylor's problem.** Sûributurigaku kenkyu. 2. Ryutairikigaku no syomondai. II. [Investigations in mathematical physics. Vol. 2. Problems of hydrodynamics. II.] Pp. 107–124. Iwanami shoten, Tokyo, 1952. 650 yen. (Japanese)

Developing a method similar to that in the paper reviewed second above, the author answers affirmatively Taylor's problem on the existence of a continuous irrotational potential flow which contains a supersonic part near a body. He further estimates by a numerical example a critical value of the Mach number at which the existence of such a flow becomes negative. *Y. Komatu* (Tokyo).

*Tamada, Kô. **On the exact solution for adiabatic irrotational flow of compressible fluid.** Sûributurigaku kenkyu. 2. Ryutairikigaku no syomondai. II. [Investigations in mathematical physics. Vol. 2. Problems of hydrodynamics. II.] Pp. 125–139. Iwanami shoten, Tokyo, 1952. 650 yen. (Japanese)

Continuing his investigations on a two-dimensional irrotational flow of a compressible fluid, the author derives a solution for the equation of motion in the hodograph plane under the supposition that the fluid obeys exactly the adiabatic law. *Y. Komatu* (Tokyo).

*Imai, Isao. **On the lift, drag and moment of an obstacle in compressible fluid flow.** Sûributurigaku kenkyu. 2. Ryutairikigaku no syomondai. II. [Investigations in mathematical physics. Vol. 2. Problems of hydrodynamics. II.] Pp. 67–106. Iwanami shoten, Tokyo, 1952. 650 yen. (Japanese)

Considering a two-dimensional uniform flow of a compressible fluid around an obstacle, the author studies its asymptotic behavior near infinity, any source or sink on the surface and any circulation being taken into account. Based on the conservation of momentum and of angular momentum, he further derives formulas for the forces and momentum exerted upon the obstacle. Similar considerations are made also in case of a discontinuous flow with a dead-water region behind the obstacle. Especially, formulas are derived which represent the lift and the drag in terms of asymptotic form of the dead water near infinity.

Y. Komatu (Tokyo).

Runyan, Harry L., and Watkins, Charles E. Considerations on the effect of wind-tunnel walls on oscillating air forces for two-dimensional subsonic compressible flow. NACA Rep. no. 1150 (1953), ii+7 pp. (1954).

The integral equation for the problem of the title is formulated by introducing the infinite set of images of the airfoil in the tunnel walls. It is shown that the resulting kernel, which appears as an infinite series of Hankel functions, becomes infinite at the resonant frequencies for standing waves between the walls. The reviewer suggests that it might be informative to exhibit the kernel as an expansion in the guided wave modes (both propagated and cutoff) of the tunnel. *J. W. Miles* (Los Angeles, Calif.).

Miles, John W. On the transonic drag of accelerated bodies. J. Aeronaut. Sci. 21, 644–645 (1954).

Cette courte note a pour but de montrer que les résultats de l'auteur [même J. 19, 280–281 (1952)] permettent d'évaluer la trainée de corps effilés en mouvement varié. Les résultats sont particulièrement simples dans le cas d'un obstacle se terminant par une pointe ou par une partie cylindrique. Les résultats sont particulièrement intéressants en régime transsonique où apparait un phénomène d'hystérésis, la valeur de la trainée n'étant pas la même en régime accéléré et en régime décéléré. Les formules simples données dans le texte ne s'appliquent qu'au régime subsonique accéléré et au régime supersonique décéléré.

P. Germain (Paris).

Lin, C. C. On a perturbation theory based on the method of characteristics. J. Math. Physics 33, 117–134 (1954).

Consider the following characteristic initial-value problem in steady irrotational supersonic plane flow. On the characteristic $\beta=0$ let $x=\alpha$, $\theta=\psi(A)-\psi(A_0)=\epsilon f(\alpha)$, and on $\alpha=0$ let $x=\beta$, $\theta=\psi(A_0)-\psi(A)=\epsilon g(\beta)$, where $y=0$ at $\alpha=\beta=0$ and $f(0)=g(0)=0$. If $(q\cos\theta, q\sin\theta)$ is the velocity and A is the Mach angle at (x, y), then

$$\psi(A)-\psi(A_0)=\int_0^q (M^2-1)^{1/2}q^{-1}dq.$$

Then

$$\theta(\alpha,\beta)=\epsilon(f(\alpha)+g(\beta)),\quad \psi(A(\alpha,\beta))-(A_0)=\epsilon(f(\alpha)-g(\beta))$$

while $x(\alpha, \beta)$ and $y(\alpha, \beta)$ satisfy

$$(1)\qquad y_\alpha=x_\alpha P(\alpha,\beta),\quad y_\beta=x_\beta Q(\alpha,\beta),$$

where

$$P(\alpha,\beta)=\tan(\theta+A),\quad Q(\alpha,\beta)=\tan(\theta-A).$$

Expand $\tan(\theta\pm A)$ in Taylor's series in θ and $A-A_0$ to obtain

$$(2)\qquad P(\alpha,\beta)=\sum_0^\infty \epsilon^n P^{(n)}(\alpha,\beta),\quad Q(\alpha,\beta)=\sum_0^\infty \epsilon^n Q^{(n)}(\alpha,\beta).$$

Write

$$(3)\qquad x(\alpha,\beta)=\sum_0^\infty \epsilon^n x^{(n)}(\alpha,\beta),\quad y(\alpha,\beta)=\sum_0^\infty \epsilon^n y^{(n)}(\alpha,\beta),$$

substitute in (1), and equate coefficients of ϵ^n to obtain systems of partial differential equations for $x^{(0)}$, $y^{(0)}$, $x^{(n)}$, $y^{(n)}$ with initial conditions $x^{(0)}(\alpha, 0)=\alpha$, $x^{(0)}(0, \beta)=\beta$, $y^{(0)}(0, 0)=0$ and $x^{(n)}(\alpha, 0)=x^{(n)}(0, \beta)=y^{(n)}(0, 0)=0$ for $n\geqq 1$. The author shows that the series (3) converge to solutions of (1) if the series (2) and their first partial derivatives with respect to α and β are uniformly bounded for all α and β. As an illustration this characteristic perturbation method is carried to completion for terms with $n=1$ in the following examples involving simple waves which cannot be treated as satisfactorily by perturbation methods with coordinates x, y as independent variables: interaction of simple waves, reflection from a rigid boundary or from a region at constant pressure, and estimation of the wave length of a plane jet. The present perturbation method differs from those of M. J. Lighthill [Philos. Mag. (7) 40, 1179–1201 (1949); these Rev. 11, 518] and G. B. Whitham [Comm. Pure Appl. Math. 5, 301–348 (1952); these Rev. 14, 330] in correcting both families of characteristics.

J. H. Giese (Ann Arbor, Mich.).

Giese, J. H. Approximate methods for computing flow fields. Comm. Pure Appl. Math. 7, 65–77 (1954).

An account is given of calculations of steady, non-viscous, non-linearized, axisymmetric supersonic flows carried out by the staff of the Ballistic Research Laboratories, Aberdeen, Maryland, on the ENIAC and ORDVAC machines. The computational method is outlined, and there is some discussion of the restrictions on both the problems which can be solved on the ENIAC and on the method of formulating them, which arise from the limited storage capacity of this machine. Some results are quoted on the magnitude of truncation errors. The list of calculations actually performed includes tables of flows about cone cylinders with cone semi-angles between 5° and 50° and Mach numbers of the incident stream between 1.3 and 7.0, calculations on the supersonic parts of a family of nozzles with plane sonic surfaces, and computations of plane nozzle shapes for use in designing cams for the control system of a wind-tunnel with a flexible throat. Mention is made of the present position with regard to some contemplated calculations involving three-dimensional rotational flow (e.g., yawing bodies of revolution).

D. C. Pack (Glasgow).

Chester, W. The diffraction and reflection of shock waves. Quart. J. Mech. Appl. Math. **7**, 57–82 (1954).

The author finds the pressure field produced on an infinite thin wedge by a plane uniform shock of arbitrary strength when the shock encounters the wedge at small incidence and an arbitrary angle of yaw, up to a limiting yaw angle. The yaw angle is restricted so that the flow behind the shock relative to the junction of the shock and the leading edge of the wedge is supersonic. Since both the angle of the wedge and the angle of incidence are small the linearization technique and method of solution developed by Lighthill [Proc. Roy. Soc. London. Ser. A. **198**, 454–470 (1949); these Rev. **11**, 478] apply almost directly and the pressure distribution on the face of the wedge is obtained in closed form. Numerical calculations indicate that the pressure distribution for a finite yaw angle is not significantly different from that corresponding to zero yaw except for values near the limiting yaw angle as defined previously. A special analysis is made of the shape and strength of the non-uniform shock wave reflected from the wedge.

P. Chiarulli (Providence, R. I.).

Alpher, Ralph A., and Rubin, Robert J. Normal reflection of shock waves from moving boundaries. J. Appl. Phys. **25**, 395–399 (1954).

By the use of standard formulas the authors have treated the one-dimensional flow resulting from the interaction of a uniform normal shock and a piston moving with constant velocity, when shock and piston move toward each other into a gas at rest. The standard interaction problem of the incident shock and the uniform shock resulting from the constant piston velocity is first treated and then the interaction of the transmitted shock and the moving wall is considered. The pressure at the wall after this last interaction is computed numerically for $\gamma=7/5$. Various limiting cases including those of infinitely strong and infinitely weak incident shock strengths are also considered.

P. Chiarulli (Providence, R. I.).

Ribner, H. S. Convection of a pattern of vorticity through a shock wave. NACA Rep. no. **1164**, ii+17 pp. (1954).

"Supersedes NACA Tech. Note no. **2864** (1953)"; these Rev. **14**, 922.

Ferrari, Carlo. On the determination of certain basic types of supersonic flow fields. NACA Tech. Memo. no. **1381**, 17 pp. (1954).

Translated from Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **7** (1949), 277–283 (1950); these Rev. **13**, 181.

Eichelbrenner, E. A. Théorie des corps élancés. O.N.E.R.A. Publ. no. **68**, 75 pp. (1954).

The wing theory of Adams and Sears [J. Aeronaut. Sci. **20**, 85–98 (1953); these Rev. **14**, 599], which carries the Munk-Jones theory of slender wings to the second order in terms of an aspect-ratio parameter, is here extended to wings whose loading at some chordwise stations is influenced by the vortex wakes of sections farther forward. Typical wings of this class are those whose trailing as well as leading edges are swept back. The report gives a complete review of the slender-wing approximations, including the extensions of it to wing-body combinations by Ward [Quart. J. Mech. Appl. Math. **2**, 75–97 (1949); these Rev. **10**, 644] and by Adams and Sears [loc. cit.], and to wings and wing-body combinations with vortex-wake influence by Legendre, Eichelbrenner and von Baranoff [O.N.E.R.A. Publ. no. **53** (1952)]. The mathematical method used is that of Fourier (Laplace) transforms for subsonic (supersonic) flows. Application is made to wings with large sweepback; the resulting formulas require numerical integration and numerical results are to be given in a subsequent paper. Numerical results are given here for certain twisted delta wings, where analytical integrations suffice. A mathematical difficulty encountered by Adams and Sears, namely the appearance of nonintegrable singularities in the second-order terms, does not occur in the cases treated here because of the particular planforms chosen.

W. R. Sears (Ithaca, N. Y.).

Clarkson, M. H. A second-order theory for three-dimensional wings in supersonic flow. Quart. J. Mech. Appl. Math. **7**, 203–221 (1954).

Wings of the class considered have supersonic edges, so that upper and lower parts of the flow field can be considered separately. Lighthill's technique [Philos. Mag. (7) **40**, 1179–1201 (1949); these Rev. **11**, 518] of rendering approximate solutions uniformly valid is used; i.e., the dependent variable (velocity potential, ϕ) and one of the independent variables (the stream-wise coordinate, x) are both expressed as Taylor series in a small parameter ϵ describing the thickness ratio or incidence. The first-order solution Φ is that of the familiar linearized theory. The second-order equation, which is linear, involves Φ on its right-hand side as well as x_1, where $x=u+\epsilon x_1+\cdots$. This equation is solved in terms involving the unknown function x_1, by means of the Riesz operator [Acta Math. **81**, 1–223 (1949); these Rev. **10**, 713; also B. Baker and E. T. Copson, The mathematical theory of Huygens' principle, 2nd ed., Oxford, 1950, pp. 54–67; for a review of the first edition see these Rev. **1**, 315].

To illustrate the method, this general solution is then applied to two-dimensional problems, in which x_1 is determined by a continuity condition. The result agrees with simple-wave theory carried to the second order. Next, the theory is applied to a wing with straight leading edge and raked (supersonic) trailing edges, rolling at constant angular velocity. Here again, a continuity condition suffices to determine x_1. In the case of a rectangular wing, however, a practical difficulty appears in the region inside the tip Mach cones. Here x_1 must be determined as in Lighthill's problem [Philos. Mag. (7) **40**, 1202–1223 (1949); these Rev. **11**, 625] to remove singularities on these Mach cones. This is done, but the resulting expressions for the second-order potential are very complicated and have not been used.

W. R. Sears (Ithaca, N. Y.).

Sewell, Geoffrey L. Theory of an accelerated slender supersonic aerofoil. Aeronaut. Quart. **5**, 52–54 (1954).

The author's solution to the title problem consists of the linearized, steady-flow solution plus perturbation potential and vorticity terms of $O[\epsilon F(x)]$ and $O[\epsilon F'(0)]$, where ϵ is the dimensionless acceleration, and $F(x)$ is the ordinate of the (symmetric) airfoil; terms of $O(\epsilon^2)$ are neglected. The reviewer notes that the resulting pressure on the airfoil (not explicitly calculated by the author) does not agree with the previous potential-flow solution (to which the author makes no reference) of Gardner and Ludloff [J. Aeronaut. Sci. **17**, 47–59 (1950); these Rev. **11**, 555; see also A. Robinson, Proc. 7th Internat. Congress Appl. Mech., 1948, v. 2, pp. 500–514; these Rev. **11**, 477; C. Roumieu, Recherche Aéronautique

1949, no. 9, 47–54 (1949); these Rev. 11, 476]. The reviewer also notes that the author's results are at variance with Cabannes' result [C. R. Acad. Sci. Paris 238, 448–449 (1954); these Rev. 15, 839] that acceleration of a two-dimensional, attached shock introduces a curvature and, consequently, a vorticity of $O[\epsilon F'(0)^2]$. These discrepancies may be due to the author's treatment of the shock conditions when $F'(0+)$ does not vanish, in which case the resulting discontinuity in $F'(x)$ at $x=0$ leads to infinite values of $F''(0)$. *J. W. Miles* (Los Angeles, Calif.).

***Heinz, C.** **Überschallströmungen um langsam pendelnde Drehkörper.** Mémoires sur la mécanique des fluides offerts à M. Dimitri P. Riabouchinsky, pp. 119–127. Publ. Sci. Tech. Ministère de l'Air, Paris, 1954. 3000 francs.

The author's previous work on the corresponding steady-flow problem [Z. Angew. Math. Mech. 33, 306–308 (1953); these Rev. 15, 367] is extended to unsteady flow. The reviewer believes that, as in the earlier paper (loc. cit.), the author's solution, in going beyond the requirements of a consistent linearized solution [cf. J. W. Miles, J. Aeronaut. Sci. 16, 378–379 (1949); Aeronaut. Quart. 2, 183–194 (1950); these Rev. 10, 755; 12, 553], is unnecessarily complex.
J. W. Miles (Los Angeles, Calif.).

Sorokin, V. S. **On steady motions in a fluid heated from below.** Akad. Nauk SSSR. Prikl. Mat. Meh. 18, 197–204 (1954). (Russian)

En se basant sur ses recherches antérieures [voir même journal 17, 39–48 (1953); ces Rev. 15, 577] l'auteur étudie les deux cas ou la valeur du gradient de température c est plus petit ou plus grand que la valeur critique c_0. Dans le premier cas, $c \leqq c_0$, l'auteur démontre qu'il n'existe aucune solution stationnaire sauf les solutions d'équilibre. Dans le second cas, $c > c_0$, on trouve, abstraction faite du cas d'équilibre, deux autres solutions stationnaires.
M. Kiveliovitch (Paris).

Žuhovickiĭ, E. M. **Application of Galerkin's method to the problem of stability of a non-uniformly heated fluid.** Akad. Nauk SSSR. Prikl. Mat. Meh. 18, 205–211 (1954). (Russian)

En négligeant les termes quadratiques des équations, l'auteur trouve, en utilisant la méthode de Galerkin, une solution approchée du problème. On applique cette solution approchée à deux exemples: 1) la surface remplie par le liquide est un cylindre vertical; 2) la surface est un cylindre horizontal. *M. Kiveliovitch* (Paris).

Bugaenko, G. A. **On free convection in an inclined cylinder.** Akad. Nauk SSSR. Prikl. Mat. Meh. 18, 212–214 (1954). (Russian)

Considérons un plan vertical $y0z$, l'axe $0z$ est dirigée suivant l'axe du cylindre et $0y$ est l'axe perpendiculaire à $0z$. On trouve aisément pour la température l'expression $T=ay+b$ (a et b étant des constantes) et la vitesse v du liquide vérifie l'équation $\Delta v=-ay$. Comme exemple d'application l'auteur étudie le cas d'un cylindre circulaire.
M. Kiveliovitch (Paris).

Drahlin, E. H. **Convection in an infinite horizontal elliptic cylinder.** Akad. Nauk SSSR. Prikl. Mat. Meh. 18, 215–218 (1954). (Russian)

En introduisant les coordonnées elliptiques dans les équations classiques d'une convection stationnaire l'auteur résoud le problème par des approximations successives en développant les solutions suivant les puissances du nombre de Grasshof. L'auteur calcule les deux premières approximations: zéro et un. *M. Kiveliovitch* (Paris).

van Mieghem, J. **Transport and production of vorticity in the atmosphere.** Tellus 6, 170–176 (1954).

The author takes the classical form of the vorticity equation and writes it in the form of an equation of balance

$$\frac{\partial Q_z}{\partial t}+\operatorname{div}(Q_z\mathbf{v}-v_z\mathbf{Q})=(\nabla p\times\nabla\alpha)_z+\operatorname{curl}_z\mathbf{F},$$

$\mathbf{Q}$ being the absolute vorticity, $\mathbf{F}$ the friction per unit mass and $\mathbf{v}$ the wind velocity. In this equation, the right-hand side represents the rate of vorticity production within a unit volume, while the term $\operatorname{div}(Q_z\mathbf{v}-\mathbf{Q}v_z)$ gives the rate of transport across the boundary of this volume. This equation is then integrated over various different regions, and in each case particular attention is paid to the total absolute vertical vorticity. By integrating over the whole atmosphere, for example, the author obtains the result that the instantaneous rate of change of total absolute vertical vorticity of the atmosphere is due to a mountain effect only. The vorticity transport is then discussed; by using the geostrophic approximation and the thermal wind equation, it can be written in the form

$$Q_z\mathbf{v}_h+\frac{g}{f}v_z\frac{\nabla_h\Theta}{\Theta},$$

where $\mathbf{v}_h$ is the horizontal wind velocity, Θ the potential temperature and f the Coriolis parameter. This gives the result that vertical vorticity is transported to regions of higher temperature if the vertical velocity v_z is positive, and vice versa. Finally, this equation is used to determine the direction of the meridional transport, and when averaged round a latitude circle this is found to be polewards in middle latitudes. *M. H. Rogers* (Urbana, Ill.).

Sretenskiĭ, L. N. **Propagation of sound in an isothermal atmosphere.** Izvestiya Akad. Nauk SSSR. Ser. Geofiz. 1954, 134–142 (1954). (Russian)

Soit $\phi(x, y, z, t)$ le potentiel des vitesses d'une particule d'un gaz oscillant; ϕ satisfait à l'équation aux dérivées partielles

$$\frac{\partial^2\phi}{\partial t^2}=c^2\Delta\phi-g\frac{\partial\phi}{\partial z},$$

où z est l'axe verticale, c la vitesse du son et g l'accélération de la pesanteur. En se bordant seulement aux solutions symétriques par rapport à l'axe des z, et en partant d'une solution particulière de la forme $Z(z)J_0(kr)$ (ou $r^2=x^2+y^2$), l'auteur trouve une solution générale en utilisant l'intégrale de Fourier-Bessel et en se donnant d'avance pour $z=0$ la dérivée normale $f(r)$. On trouve de cette façon le potentiel du champ acoustique. Deux cas sont à considérer suivant le signe de $g^2-4c^2\sigma^2$ où σ est la fréquence des oscillations. L'auteur déduit ensuite les expressions asymptotiques valables pour des distances éloignées de la source.
M. Kiveliovitch (Paris).

Elasticity, Plasticity

Berio, Angelo. **Deformazioni finite dei corpi elastici.** Rend. Sem. Mat. Fis. Milano 23 (1952), 164–182 (1953).

An inexact exposition of classical work on finite elastic strain, citing Italian sources. [The author's qualification of

one of Murnaghan's forms of the stress-strain relations [Amer. J. Math. 59, 235–260 (1937), p. 248] as "approximate" results from his own false eq. (28), which is based on a confusion of material and spatial co-ordinates derived from some of his sources.] *C. Truesdell.*

Swann, W. F. G. Stress and strain. Discussion of Dr. Philippoff's paper. J. Appl. Phys. 25, 1108–1109 (1954).

The author discusses one-dimensional visco-elastic theories. His object is to write down a relation for the stress which has an assigned steady-state form and which for the case of sinusoidal oscillation reduces to a linear combination of strain and strain rate with coefficients which are assigned functions of frequency. His result is a non-linear formula involving derivatives of the strain up to the sixth.

C. Truesdell (Bloomington, Ind.).

Courbon, Jean. Insuffisance des solutions données à certains problèmes d'élasticité plane. Ann. Ponts Chaussées 124, 475–483 (1954).

The author considers the following three problems of two-dimensional elasticity. I. Distributions of stresses in a dihedron of specific gravity π suspended along its edge, which is supposed to be held horizontally. II. Distribution of stresses in an elastic half-plane which is submitted to the action of a normal force at one point of its boundary. III. The penetration of two elastic cylindrical bodies in each other along their generatrices, supposed to be pressed against each other. He observes that the known theoretical solutions of these three problems do not agree at all with experiment and explains this by the fact that the corresponding displacements in the bodies are no longer infinitesimal, in contradiction with the fundamental hypothesis of linear elasticity. *C. Arf* (Istanbul).

Dean, W. R., and Harris, G. Z. The Green's function of an elastic plate. Mathematika 1, 18–23 (1954).

In this paper a simple expression is obtained for the small transverse displacement w of a thin isotropic plate due to a transverse force applied at an arbitrary point of the plate. The plate is bounded internally by a parabola along which it is clamped. Rectangular axes are taken in the plane of the plate so that the boundary is given by $y^2=4\alpha^2(x^2+\alpha^2)$, $\alpha \geqq 0$; and the transverse force is applied at some point (x_0, y_0) of the plate. The displacement w at any point (x, y) of the plate is a biharmonic function which is such that, along the boundary, w together with its normal derivative is zero. It is also a symmetrical function of the coordinate pairs (x, y) and (x_0, y_0). It is thus the Green's function associated with the differential equation and the boundary conditions. Approximate expressions for w are derived for points near the boundary and for points at a great distance from the boundary. The particular case in which the parabola degenerates into a straight line is considered and from this case the known Green's function for a semi-infinite plate is derived as a limit. *R. M. Morris* (Cardiff).

Kammerer, A. Sur l'influence de la courbure de la ligne moyenne des solides en résistance des matériaux. Bull. Soc. Franç. Méc. 4, no. 13, 35–44 (1954).

Some analysis is given to determine the small deflections of a thin elastic beam under known forces. The cross-section of the beam is arbitrary, and the neutral axis of the beam is a general twisted curve. Some particular examples are treated. *H. G. Hopkins* (Fort Halstead).

Ling, Chih-Bing, and Lee, Ti-Chiang. Stress systems in an infinite cylinder. Ann. Acad. Sinica, Taipei 1, 517–526 (1954).

This paper contains particular solutions of the field equations in the classical theory of elasticity, appropriate to the problem of torsionless axisymmetry for a circular cylinder of infinite length. The solutions are generated by means of Love's stress function and are in Fourier integral form. The problems presented by a cylinder subjected to a circumferential normal or tangential line load are considered in detail. No reference is made to the extensive previous literature on this subject [see, e.g., I. N. Sneddon, Fourier transforms, McGraw-Hill, New York, 1951; these Rev. 13, 29]. *E. Sternberg* (Chicago, Ill.).

Ling, Chih-Bing, and Yang, Kuo-Liang. On symmetrical strain in solids of revolution in curvilinear coordinates. Ann. Acad. Sinica, Taipei 1, 507–516 (1954).

Explicit formulas are given, in general axisymmetric orthogonal curvilinear coordinates, for the components of displacement and stress generated by Love's stress function for the torsion-free axisymmetric problem in elasticity theory. *E. Sternberg* (Chicago, Ill.).

Higuchi, Masakazu. On the axisymmetric boundary value problems in the transversely isotropic elasticity theory. Rep. Res. Inst. Appl. Mech. Kyushu Univ. 3, 143–145 (1954).

In this paper dealing with general solutions of the equations of linear elasticity the author attempts to do more than to express these solutions in terms of solutions to other differential equations. Such a general solution is used to construct solutions for a class of stress and displacement boundary-value problems for transversely isotropic materials. The author indicates a number of types of problems which can be solved by this method and presents the solutions for specified stress and displacement boundary conditions for a semi-infinite body bounded by a plane, assuming axial symmetry. *J. L. Ericksen* (Washington, D. C.).

Morgan, Antony J. A. Stress distributions in semi-infinite solids of revolution. Z. Angew. Math. Physik 5, 330–341 (1954).

The author obtains solutions of the equations of linear elasticity for semi-infinite bodies which are bounded by two concentric cones with common apex, the applied tractions being proportional to a power of the distance from the apex. The technique used is to reduce the relevant partial differential equations to an ordinary differential equation, using a theorem proved in Quart. J. Math., Oxford Ser. (2) 3, 250–259 (1952) [these Rev. 15, 37]. The general solution of the latter is obtained in terms of Legendre functions.

J. L. Ericksen (Washington, D. C.).

Odqvist, F. K. G. Theorie der elastischen Ringe starker Krümmung. Ing.-Arch. 22, 98–107 (1954).

An approximate (engineering) theory is given of the elastic deformation of a thin ring under general forces. The ring is of arbitrary (but constant) cross-section. Application is made to some problems involving concentrated forces and moments. *H. G. Hopkins* (Fort Halstead).

Grigolyuk, È. I. On the strength and stability of cylindrical bimetallic shells. Akad. Nauk SSSR. Inženernyĭ 16, 119–148 (1953). (Russian)

A bimetallic shell is a shell consisting of two layers, each having different mechanical properties. The surfaces of

contact are rigidly united, they cannot slide one against the other, they must move and be deformed together. The Love-Kirchhoff hypothesis for homogenous shells can be modified for two-layer shells. In this case the contact surface instead of the middle surface is used as the initial surface. The applicability of Love-Kirchhoff theory for two-layer shells depends on the ratio of elastic moduli. If the ratio is less than approximately 10–15, the theory is applicable. If more than 10–15, torsional deformations cannot be neglected, and the theory is not applicable.

The author considers a two-layer cylindrical shell where Love-Kirchhoff theory is applicable. The loads and the heating stresses, which are also taken into account, are always axially symmetric. The author derives the basic equations and then solves them for shells of infinite length, for shells of finite length with various boundary conditions. The stability of shells under axial compression is investigated at static and dynamic conditions. At dynamic conditions the compression has a periodic component and the author finds the critical frequency. At the end the author gives a short extension of his theory for many-layered shells. Numerical tables are appended. *T. Leser.*

McFadden, J. A. Radial vibrations of thick-walled hollow cylinders. J. Acoust. Soc. Amer. **26**, 714–715 (1954).

Approximate formulas are given for the natural wavelengths associated with the free radial vibration of a thick-walled, infinitely long, hollow cylinder.

From the author's summary.

Adem, Julián. On the axially-symmetric steady wave propagation in elastic circular rods. Quart. Appl. Math. **12**, 261–275 (1954).

In the first part of this report we find an exact solution for the problem of steady wave propagation in an isotropic, elastic circular bar of infinite length, free of stress on its lateral surface and loaded by a harmonic body force, parallel to its axis, whose amplitude is a Dirac delta function. The solution of this problem gives at the same time the solution for a semi-infinite bar with a prescribed load on its boundary plane. In the second part, we find a solution for the semi-infinite bar in which the conditions at the boundary plane are prescribed in terms of functions which give implicitly the stresses and displacements. Finally, using a frequency of interest in current ultrasonic experimental work, we develop a numerical example and compare the result with the case of a low frequency. (Author's summary).

R. M. Morris (Cardiff).

Iguchi, S. Die Eigenschwingungen und Klangfiguren der vierseitig freien rechteckigen Platte. Ing.-Arch. **21**, 303–322 (1953).

In this impressive work the author has treated, from the analytical, numerical and experimental standpoints, the problem of the determination of frequencies and mode shapes in free vibration of an isotropic, rectangular plate with all edges free. According to a recent bibliographical survey on the vibrations of rectangular plates by R. F. S. Hearmon [J. Appl. Mech. **19**, 402–403 (1952)] this problem has received significant attention in only one prior paper, that of W. Ritz [Ann. Physik (4) **28**, 737–786 (1909)] in which 35 frequencies and mode shapes for the free-sided square plate were calculated by the Ritz method. Using the mode shapes of the free-free beam as approximating functions for the plate mode shapes, the Ritz method obtains upper bounds for the frequencies.

In the first section of this paper, an exact solution for the vibration of free-sided rectangular plates is presented as a sum of two infinite expansions, each term of which individually satisfies the differential equation of motion. Determining the coefficients in these expansions to satisfy boundary conditions leads to an infinite system of linear equations in the coefficients; the determinant of this system is the characteristic equation for the plate frequencies. To solve this equation for the general rectangle is, however, exceedingly complicated, and therefore numerical calculations are done for the square plate only. Further simplification is accomplished by ordering the vibrations on the square into three classes, characterized with respect to the center-lines by total symmetry, total anti-symmetry and mixed symmetry and anti-symmetry. The first two classes are further subdivided each into two families in which total symmetry or total anti-symmetry with respect to the diagonals is assumed. Substantially the same grouping was used by Ritz. There are then computed and tabulated, from the first four or five rows and columns of the characteristic determinant in each case, the values for the first four or five frequencies in each family and, along with each frequency, values for the first four or five coefficients in the expansion for the mode shape. A sketch of the nodal line configurations (Klangfiguren) accompanies each frequency in the tables. For representative frequencies in each family, a detailed map of level lines and cross-section contours of the mode shape is given. Numerical values are given to six significant figures, although accuracy of results and estimates of error are not discussed; in all 23 frequencies are computed for Poisson's ratio $=0.3$. Frequencies of the third class are shown to have an infinite set of possible mode shapes which are formed as linear combinations of the two basic mode shapes for that frequency.

In the last section of his paper, the author reports on his experiments in producing nodal line patterns by "sand figures" using square glass plates of sides 20 to 28 cm. and from 2 to 3 mm. thick. Twenty photographs of sand figures are presented, most of which show excellent agreement with the analytical patterns. Of particular interest is the experimental demonstration of the analomous behavior of third class vibrations. Reviewer's note: It is regrettable that the author did not choose to include a detailed comparison of his results with those of Ritz. Although Ritz's computations used Poisson's ratio as 0.225, he included for the lowest frequencies a correction term to account for different Poisson's ratios. The reviewer has computed the Ritz frequencies so corrected; the largest difference found was in the seventh mode, where Ritz's result was about 1% less than that of the author. *W. Nachbar* (Seattle, Wash.).

Scott, E. J. Wave propagation in a visco-elastic medium. Quart. Appl. Math. **12**, 300–306 (1954).

The author considers a medium confined between two parallel planes whose deformation in the direction perpendicular to the planes can be represented by the motion of a one-dimensional model composed of n units coupled in parallel, each unit consisting of a spring and a dashpot connected in series. The deformation is then described by an integro-differential equation which is treated by the method of Laplace transform. The case where one plane remains at rest while the other is subject to an impulse velocity ($v=0$ for $t<0$ and $v=V_0$ for $t>0$) is considered. In the case $n=2$ an explicit solution is given. The result is the superposition of a uniform compression and a damped oscillation.

W. Noll (Berlin).

***Geiringer, Hilda. Remark on the integration of the stress equations in plane plasticity.** Mémoires sur la mécanique des fluides offerts à M. Dimitri P. Riabouchinsky, pp. 85–87. Publ. Sci. Tech. Ministère de l'Air, Paris, 1954. 3000 francs.

For an ideal plastic body under plain strain, it is known that the radii of curvature, R_ξ, R_η, of the slip lines, $\eta=$const. and $\xi=$const., respectively, satisfy the partial differential equations

$$\frac{\partial^2 R_\xi}{\partial\eta\partial\xi}=R_\xi,\quad \frac{\partial^2 R_\eta}{\partial\eta\partial\xi}=R_\eta. \tag{1}$$

In this paper, the author points out that, in addition to R_ξ, R_η, another pair of functions satisfy (1). These are the functions

$$X=x\cos(\eta-\xi)+y\sin(\eta-\xi),\qquad Y=-x\sin(\eta-\xi)+y\cos(\eta-\xi), \tag{2}$$

where x, y are the Cartesian orthogonal coordinates of a point. *N. Coburn* (Ann Arbor, Mich.).

Handelman, G. H., and Warner, W. H. Loading paths and the incremental strain law. J. Math. Physics **33**, 157–164 (1954).

In general, the correct analysis of a problem of plastic deformation involves the use of an incremental rather than a total stress-strain law. However, in spite of this fact, situations often do arise in which the predictions of the analysis are not greatly dependent upon the type of stress-strain law that is chosen. This paper gives a proof, subject to certain stated assumptions, that such predictions can only be identical for certain (and often proportional) loading paths. A treatment is also given of the condition for the total plastic work to be independent of the loading path. As the authors observe, the simplicity of a total stress-strain law renders it preferable on mathematical grounds to an incremental stress-strain law. Therefore an investigation leading to an estimate of the error involved in any particular case following the replacement of the latter law by the former would be of great use in practical problems, but this question is not further discussed. *H. G. Hopkins.*

MATHEMATICAL PHYSICS

Whyte, L. L. A dimensionless physics? British J. Philos. Sci. **5**, 1–17 (1954).

The author expounds the thesis that atomic theory should be expressed in units which are dimensionless in length, time, and mass. Though the discussion is non-mathematical, the documentation is interesting. *G. Birkhoff.*

Optics, Electromagnetic Theory

Dossier, Brigitte. Recherches sur l'apodisation des images optiques. III. Rev. Optique **33**, 267–297 (1954).

Concluding part of the investigation of which Parts I and II were reviewed earlier [Rev. Optique **33**, 2–111, 147–178 (1954); these Rev. **15**, 913]. Rectangular and circular screens represented by 2, 3 and 4 terms of appropriate functions (trigonometric or Dini-Bessel) are considered. It is shown that one may obtain circular screens which are represented by two terms and have a degree of apodization between 900 and 1. For screens represented by 3 terms, the loss of light may be halved for a given degree of apodization. The more important cases are illustrated by diagrams of diffraction patterns and are compared with experiment. *E. Wolf* (Manchester).

Kay, Irvin, and Keller, Joseph B. Asymptotic evaluation of the field at a caustic. J. Appl. Phys. **25**, 876–883 (1954).

The method of R. K. Luneberg, for the asymptotic development (with respect to the wave number k) of steady-state electromagnetic fields breaks down at points in the field which lie on the geometrical caustic. This paper is concerned with the true asymptotic behaviour at such points. At ordinary points the leading term of the asymptotic series (the term representing the geometrical-optics field) is proportional to k^{-1}. At points on the caustics, it is found that this is no longer the case. The asymptotic expansion is of a different character then, the leading terms being proportional to $k^{1-1/n}$, where n is an integer greater than 2. The results are applied to the field of a plane wave falling onto a parabolic and a circular cylinder respectively. *E. Wolf* (Manchester).

Steel, W. H. Calcul de la répartition de la lumière dans l'image d'une ligne. Rev. Optique **31**, 334–340 (1952).

The light intensity on a focal plane of an aberration-free optical system due to a line source is given by the well known Struve function $H_1(2x)x^{-1}$, where x is the distance from the axis of the optical system. The author derives the same result by means of Fourier transformation. The intensity is also expressed in a series summed over the argument of a Bessel function of order one. As a special case, the intensity along the axis is calculated from the known properties of Lommel functions and also by a direct application of Fourier transformation. The intensity calculated by the latter method is given by an integral involving the square of Fresnel integrals over the interval $(0, 2\pi)$, whereas by using the first method one obtains an infinite series in the axial coordinate. The intensity curve due to a point source vanishes at certain points on the axis, while for a line source the minima of the intensity curve remain positive. *N. Chako* (New York, N. Y.).

Steel, W. H. Etude des effets combinés des aberrations et d'une obturation centrale de la pupille sur le contraste des images optiques. Application aux objectifs de microscope à miroirs. Rev. Optique **32**, 4–26, 143–178, 269–306 (1953).

In the first part of this paper the author investigates the effect of an annular aperture on the light distribution on a Gaussian plane of an ideal optical system due to a point and a line source and also for an infinite net (resolution chart) of frequency 2ω. A new definition of the quality of the image is introduced (def. A), which is based on a luminous point, instead of the usual one based for a dark point on a uniform bright background (def. B). The intensity at a point $(x, 0)$ in the geometrical focal plane due to a line source is given in terms of two Struve functions $H_1(2x)x^{-1}$ and $H_1(2x)x^{-1}p$ and an infinite series in $(1+p)x$ with coefficients depending on the complete elliptic functions $E(p)$ and $K(p)$, p being the radius of the central obstruction of the exit-pupil. The

central intensity, $x=0$, for a resolution chart of frequency 2ω is calculated for both a full circular pupil and an annular aperture. For values of p which lie within the tolerances of def. A, the contrast of the image is found to be better when compared with that obtained for a full aperture with aberrations present for all values of ω between 0 and 1, whereas according to def. B the contrast is worse for $\omega<0.55$. The intensity along the axis due to a point and a line source is also evaluated [see the preceding review]. [The general case of the effect of an annular aperture due to point sources on the three-dimensional light distribution near the Gaussian image plane has been thoroughly investigated by Linfoot and Wolf in Monthly Not. Roy. Astr. Soc. **112**, 452–469 (1952); Proc. Phys. Soc. Sect. B. **66**, 145–149 (1953); and for a system of annular apertures by Boivin in J. Opt. Soc. Amer. **42**, 60–64 (1952); these Rev. **13**, 801.] The first part concludes with a discussion on the light distribution from coherent sources.

The second part generalizes the above results by the introduction of aberrations. Here the author follows the procedure first given by Zernike and Nijboer [Nijboer, Physica **10**, 679–692 (1943); **13**, 605–620 (1947); these Rev. **6**, 108; **11**, 141; **12**, 305] by introducing a set of polynomials which reduce to Zernike polynomials when $p=0$. The intensity is then expressed (making the usual approximations) in terms of individual aberrations on account of the orthogonality of the polynomials entering in the expansion of the deviation function of the wave front due to the presence of aberrations. The effect of each individual aberration and combinations of them on the quality of the image is studied in detail. The intensity at the geometrical focal point is evaluated by means of Fourier transformation, and the quality of the image is given up to p^8. The effect of aberrations on the image quality for a resolution chart is treated in a similar manner, and numerical values of the aberration coefficients for different values of ω for $p=0$ and $p=0.2$ are tabulated. Finally, the case of a circular disc is analyzed in detail. In this case the quality of the image is found to be independent of the radius of the disc if the tolerances, i.e. the values of p, obtained from def. A are used, while the quality of the image depends on the radius of the disc if those allowed by def. B are used.

Part three contains a discussion on the merit of the two definitions A and B for extended objects. It is found that both theory and experiment favor the tolerances of def. A when extended objects are imaged, such as a line, a disc, and resolution charts. The last part deals with the effect of a central obstruction for a system of reflecting mirrors (reflecting microscope) based on geometrical optics. The following combination of mirrors (spherical) are studied: (a) objectives formed by two concentric mirrors, (b) three mirrors, and (c) four mirrors not necessarily concentric. The main result of this section is that for a system formed by two concentric mirrors the effect of spherical aberration and of central obstruction do not compensate each other in order to give a good quality image. However, if three or four mirrors are used the obstruction can be reduced without introducing much spherical aberration, and in this way one could obtain an improved image. *N. Chako.*

Wolf, E. A macroscopic theory of interference and diffraction of light from finite sources. I. Fields with a narrow spectral range. Proc. Roy. Soc. London. Ser. A. **225**, 96–111 (1954).

The author presents a systematic study of interference and diffraction in stationary optical fields produced by finite sources that emit light within a finite frequency range. A Huygens principle involving only observable quantities is formulated. This principle expresses the intensity at any field point in terms of an integral taken twice independently over an arbitrary surface, the integral involving the intensity distribution over the surface and the values of a certain correlation factor [degree of coherence; cf. Zernike, Physica **5**, 785–795 (1938)]. Various properties of the author's correlation and coherence formulas are commented upon in the light of experiments and in relation to previous results of Zernike, van Cittert, Hopkins, and others.

C. J. Bouwkamp (Eindhoven).

Bouwkamp, C. J. Diffraction theory. Reports on Progress in Physics **17**, 35–100 (1954).

This is a critical survey of recent progress in diffraction theory, especially for plane obstacles, with reference to over 500 papers published since 1940. Most of the material from the author's previous, but less comprehensive (61 references) survey [Mathematics Research Group, Washington Square College of Arts and Science, New York University, Research Rep. No. **EM-50** (1953); these Rev. **14**, 1148] is included. The material is divided into twelve sections, as follows: §1 is an introductory discussion of the types of diffraction problems to be covered and methods of solution. Reference is made to other recent survey papers on diffraction problems excluded by the author. §2 formulates the scalar problems for plane obstacles. A careful discussion of sufficient conditions for the unique characterization of the solution is given. Starting from the well known Rayleigh formulae, integral and differential-integral equations are established, the non-integrable kernels and divergent integrals of many earlier writers being avoided. Fourier (spectral representation) type formulations also are noted. §3 formulates the electromagnetic problems for plane obstacles with particular emphasis on the much debated (and now resolved) question of edge singularities and on Babinet's principle. §4 deals with Kirchhoff's theory for scalar problems and its various modifications. §5 discusses approximate methods for electromagnetic problems, especially those based on or related to Kirchhoff's scalar theory. §6 presents Schwinger's variational formulations for scalar problems. A resolution of the paradox [H. Levine and J. Schwinger, Physical Rev. (2) **75**, 1423–1432 (1949); these Rev. **10**, 764] presented by the k^2 behaviour (correct: k^4) of the low-frequency scattering cross-section when the variational formulation is expressed in terms of the solution on the screen (rather than in the aperture) is suggested by the author, viz., that the correct coefficient of the k^2 term should vanish identically. §7 deals with the Wiener-Hopf technique. §8 discusses scalar diffraction by small apertures and narrow slits on the basis of Rayleigh's approximations and extensions thereof. (The author misquotes Rayleigh as to the applicability of the scalar results to electromagnetic problems; the adjective sufficient should be replaced by insufficient.) §9 takes up the specific problem of electromagnetic diffraction by a circular obstacle, and errors in the work of Bethe and Copson are corrected. §10 covers the solution of slit (or ribbon) and circular aperture (or disk) problems by expansions in Mathieu and spheroidal wave functions, respectively. §11 reviews work on diffraction by spherical and cylindrical obstacles. §12 briefly surveys problems not incorporated in §1–11. The reviewer regrets that no mention is made of the many important ideas advanced by Bateman [Electrical and optical wave motion, Cambridge, 1915],

albeit the author explicitly restricts his coverage to post 1940 developments. In summary, the author has produced an authoritative and comprehensive survey that should be read by every serious worker in diffraction theory.

J. W. Miles (Los Angeles, Calif.).

Crysdale, J. H. Comments on "Diffraction of electromagnetic waves by an aperture in a large screen". J. Appl. Phys. **25**, 269–270 (1954).

Bekefi's approximate solution [same J. **24**, 1123–1130 (1953); these Rev. **15**, 271] to the problem mentioned in the title is derived "from a simple and straightforward argument based on the physical aspects of the problem". [For a broad discussion of the subject, see the paper reviewed above, in particular §5.] *C. J. Bouwkamp.*

Ament, W. S. Application of a Wiener-Hopf technique to certain diffraction problems. Naval Research Laboratory, Washington, D. C., NRL Rep. **4334**. ii+23 pp. (1954).

The factoring method required to decompose the Fourier transform of the kernel of an integral equation of the Wiener-Hopf type can be difficult at times. The author suggests that a transformation to another complex plane may help matters and illustrates it with application to some problems in diffraction theory. *A. Heins.*

v. Fragstein, C. Über die Gültigkeit des Sommerfeld-Pfrangschen Reziprozitätstheorem in absorbierenden Medien. Optik **11**, 301–311 (1954).

It is shown that the reciprocity theorem of Sommerfeld and Pfrang in the theory of radiation from antennae is not valid in an absorbing medium. It is pointed out that Sommerfeld's proof does not cover this case, because the expressions for the Hertz dipole used by him do not apply when the medium is absorbing. A modified formulation of the theorem is proposed. *E. Wolf* (Manchester).

Pircher, Georges. Le théorème de réciprocité appliqué aux jonctions de guides d'ondes polarisées elliptiquement. C. R. Acad. Sci. Paris **239**, 694–696 (1954).

The author applies the reciprocity theorem to an arbitrary junction of waveguides supporting elliptically polarized waves. He performs the analysis for two equivalent descriptions of an elliptically polarized wave. One of these is based on a decomposition of the wave into two linearly polarized orthogonal components and the other on a decomposition into two oppositely sensed rotating components.

C. H. Papas (Pasadena, Calif.).

Rubinowicz, A. Über die Fortpflanzung unstetiger elektromagnetischer Signale in Wellenleitern. Acta Phys. Polonica **13**, 115–133 (1954). (Russian summary)

The article contains a rather complete discussion of nonperiodic propagation in wave guides. The technique is based on an appeal to solutions of the de Broglie equation found by the author [same Acta **10**, 79–86 (1950); these Rev. **15**, 919]. Fourier or Laplace transformation in the time domain is not applied, although the results, as noted by the referee, are equivalent to those obtained by J. D. Pearson [Quart. J. Mech. Appl. Math. **6**, 313–335 (1953); these Rev. **15**, 266] for the scalar (sound) case. A full treatment of the electromagnetic case, by use of operational methods, is also given in the thesis of M. Cerrillo, who has extended the method of stationary phase for the evaluation of some of the difficult integrals which occur [Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Mass., Rep. no. **33** (1947)]. *W. K. Saunders.*

Lurye, Jerome. The electromagnetic field of a dipole over a dielectric slab on a finitely conducting ground plane. Division of Electromagnetic Research, Institute of Mathematical Sciences, New York University, Research Rep. no. **EM-65**, i+48 pp. (1954).

The report is an attempt to generalize an earlier work on the guiding of waves by coated conductors. A lossless dielectric slab is sandwiched between a semi-infinite free-space region and a semi-infinite lossy medium, in which waves are set up by an electric dipole situated in free space with axis perpendicular to the slab. A novel feature of the author's treatment is that the effect of the lossy medium is approximately taken into account by assuming a boundary condition $\partial u/\partial n = \beta u$ at the interface of slab and lossy medium, which considerably simplifies the calculations. The constant β depends on the wave number of the lossy medium. The resulting boundary-value problem is solved in the conventional way involving Bessel-function integrals. The wave functions in free space and in the slab are evaluated asymptotically, that is, for observation points whose horizontal distance from the dipole is at least several free-space wavelengths. The mathematical technique is that of van der Waerden [Appl. Sci. Research B. **2**, 33–45 (1951); these Rev. **12**, 808]. The corresponding expressions for the electric and magnetic fields are derived. The guided modes in the slab are discussed in detail. The connection between the lowest-order mode and the Sommerfeld surface-wave is established. *C. J. Bouwkamp* (Eindhoven).

Twersky, V. Certain transmission and reflection theorems. J. Appl. Phys. **25**, 859–862 (1954).

The author first considers the scattering of a plane wave by a cylinder of arbitrary cross-section and arbitrary dielectric properties. The absorption cross-section is defined as the time-averaged energy flow into unit length of cylinder divided by the time-averaged flux per unit area of incident wave. The scattering cross-section is defined as the time-averaged scattered power per unit length of cylinder, divided by the time-averaged incident flux density. The total cross-section is the sum of these two and is the energy flux the cylinder obtains from the incident wave and dissipates as heat and scattered radiation. An expression for the total cross-section is derived. This is then applied to the transmission of a plane wave through a planar uniform distribution of identical parallel cylinders, next, to the analogous problem of the reflection of a plane wave incident on a surface composed of an arbitrary cylindrical protuberance on a smooth perfectly reflecting plane. The total reflection cross-section is evaluated in this case. In analogy to the scattering case a theorem is derived, stating that the total energy cross-section of an arbitrary boss on a perfectly reflecting plane is proportional to the real part of the scattering amplitude evaluated in the direction of specular reflection of the plane. *M. J. O. Strutt* (Zurich).

Parzen, Philip. Electromagnetic wave propagation in bounded electron beams. Quart. Appl. Math. **12**, 309–312 (1954).

In this note the author applies the uniqueness theorem to an electron beam in a waveguide with an infinitely strong magnetic focusing field. He restricts his discussion from the start to the case of harmonic time dependence. And, there-

fore, there still remain some pertinent questions that should be settled. *C. H. Papas* (Pasadena, Calif.).

Nardini, Renato. Soluzione di un problema al contorno della magneto-idrodinamica. Ann. Mat. Pura Appl. (4) 35, 269–290 (1953).

This paper is concerned with the solution of a boundary-value problem in the theory of magneto-hydrodynamic waves in a perfect incompressible fluid. If the fluid has density δ, permeability μ and conductivity γ (all constant), the phenomenon is governed by the differential equations

$$\frac{\partial \mathbf{H}}{\partial t}=\frac{1}{\mu\gamma}\nabla^2\mathbf{H}+\operatorname{curl}(\mathbf{v}\times\mathbf{H}), \qquad \operatorname{div}\mathbf{H}=0,$$

$$\frac{\partial \mathbf{v}}{\partial t}=-\frac{1}{\delta}\operatorname{grad}p-\frac{\mu}{\delta}\mathbf{H}\times\operatorname{curl}\mathbf{H}, \qquad \operatorname{div}\mathbf{v}=0,$$

where $\mathbf{v}$ is the velocity vector, p the pressure, and $\mathbf{H}$ the magnetic force vector. Using rectangular cartesian coordinates, the author considers the case when p, $\mathbf{v}$, $\mathbf{H}$ depend only on z and t, and when the fluid fills the half-space $z>0$. The initial conditions are that $v_z=0$, $H_x=H_y=0$, $H_z=$constant. The boundary conditions on $z=0$ are $v_z=0$, H_x and H_y given. Moreover, p takes an assigned value $F(t)$ when $z=z$.

It turns out that H_z remains constant and that v_z is everywhere zero, so that the motion is parallel to the plane $z=0$. The differential equations reduce to

$$\frac{\partial H_i}{\partial t}=\frac{1}{\mu\gamma}\frac{\partial^2 H_i}{\partial z^2}+H_z\frac{\partial v_i}{\partial z}, \quad \frac{\partial v_i}{\partial t}=\mu\frac{H_z}{\delta}\frac{\partial H_i}{\partial t},$$

where i indicates x or y. Thus we can consider the determination of the pairs (v_x, H_x) and (v_y, H_y) separately. When H_x and H_y are found, the pressure is given by

$$\frac{\partial p}{\partial t}=-\tfrac{1}{2}\mu\frac{\partial}{\partial z}[H_x^2+H_y^2].$$

The conditions under which the solutions are sought are $H_i(z,0)=0$, $v_i(z,0)=v_0{}^i(z)$, $H_i(0,t)=G_i(t)$, $H_i(\infty,t)=0$. It follows that, when $t=0$,

$$\frac{\partial H_i}{\partial t}=H_z\frac{\partial v_0{}^i}{\partial z}$$

so that $\partial H_i/\partial t$ is known when $t=0$. Dropping the label i, we find that H satisfies a third-order equation

$$\frac{\partial^3 H}{\partial t^3}=\frac{1}{\mu\gamma}\frac{\partial^3 H}{\partial t\partial z^2}+\frac{\mu}{\delta}H_z^2\frac{\partial^3 H}{\partial z^2}$$

with constant coefficients. The particular solution of this when $v_0(z)\equiv 0$ is found formally by the Laplace transform with regard to the variable t. *E. T. Copson.*

Rumsey, V. H. Reaction concept in electromagnetic theory. Physical Rev. (2) 94, 1483–1491 (1954).

The concept of reaction is defined as an abstract complex number, obtained by integrating over all space the product of the electric field strength generated by an electric source b and a second electric source a minus the product of the analogous magnetic quantities. In the simple case of two electric point sources this reaction essentially gives the force exerted by the one source on the other. This concept of reaction is then used to obtain formulas for scattering and transmission coefficients and also for aperture impedances. An approximate solution of such problems is obtained by replacing the correct source with an approximate source which is adjusted so as to give the correct reaction with certain test sources. This approach to these problems is shown to be more universal than the variational approach. This is illustrated by the problem of scattering by a dielectric body. *M. J. O. Strutt* (Zurich).

Slichter L. B. An electromagnetic interpretation problem for the sphere. Proc. Roy. Soc. London. Ser. A. 214, 356–370 (1952).

The electric properties (dielectric constant ϵ, magnetic permeability μ and conductivity σ) within an isotropic sphere of radius ρ_1 in free space are assumed to be continuous unknown functions of the radial distance ρ alone. It is the purpose of this paper to determine these unknown functions $\epsilon(\rho)$, $\mu(\rho)$, $\sigma(\rho)$ from observations of the resultant magnetic field on its surface which is induced by an external source consisting of a symmetrical current system outside the sphere. The unknown variation of the electric properties is determined by a Taylor series. *F. Oberhettinger.*

Epstein, Bernard. Determination of coefficients of capacitance of regions bounded by collinear slits and of related regions. Division of Electromagnetic Research, Institute of Mathematical Sciences, New York University, Research Rep. no. BR-7, i+12 pp. (1954).

The author derives several formulas for the coefficients of capacitance p_{ik} $(i,k=1,2,\cdots,m)$ for a slit domain D consisting of a finite number m of collinear slits cut in a plane, and he applies the most convenient of these to the calculation of the p_{ik}'s of a bi-filar shielded cable. His method of attack consists of conformally mapping the cable domain, which is a region bounded externally by one circle and internally by two others, into a slit domain D for which he calculates the p_{ik}'s, and then of using the conformal invariance property of the p_{ik}'s to obtain the coefficients of capacitance for the cable domain. *C. H. Papas.*

***Snow, Chester. Formulas for computing capacitance and inductance.** National Bureau of Standards Circular 544. U. S. Government Printing Office, Washington, D. C., 1954. 69 pp. $.40.

The author presents a valuable collection of explicit formulas for capacitance between two conductors including spheres, spheroids and toroids; for inductance, both self- and mutual, of circular filaments, solenoids and parallel long conductors; and for skin effect in concentric cable as well as proximity effect for parallel wires. For the inductance formulas, their general basis of derivation is given, including the vector potential theory and the concept of Maxwell's mean geometric distance. Explicit derivations in the more difficult coordinate systems are given in the last section. *E. Weber* (Brooklyn, N. Y.).

Dirac, P. A. M. A new classical theory of electrons. III. Proc. Roy. Soc. London. Ser. A. 223, 438–445 (1954).

Two previous papers [same Proc. 209, 291–296 (1951); 212, 330–339 (1952); these Rev. 13, 893; 15, 228] are extended to the case of a finite number of interpenetrating streams of electricity moving under the Lorentz law and interacting only via the electromagnetic field produced by the streams according to Maxwell's equations. In anticipation of application of the theory to quantum mechanics, the equations governing the motion of the streams are expressed in Lagrangian, Hamiltonian and Poisson bracket form. *A. J. Coleman* (Toronto, Ont.).

Marx, G. The equation of motion of the magnetic dipole. Bull. Acad. Polon. Sci. Cl. III. **2**, 219–224 (1954).

The formal solution of Maxwell's inhomogeneous field equations containing electric and magnetic momentum densities is given in accordance with the Einstein-Infeld-Wallace method [Infeld and Wallace, Physical Rev. (2) **57**, 797–806 (1940); these Rev. **1**, 274]; the resultant equation of motion for the magnetic dipole reduces to the static case for the dipole at rest, but differs from previous results [H. J. Bhabha and H. C. Corben, Proc. Roy. Soc. London. Ser. A. **178**, 273–314 (1941); these Rev. **3**, 158] in the term containing the acceleration. *E. Weber.*

***Weber, Ernst. Linear transient analysis. Vol. I. Lumped-parameter two-terminal networks.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1954. xiv+348 pp. $7.50.

This volume admirably fills an important gap in the text material available for the instruction of engineers and applied mathematicians in advanced circuit theory. It is concerned with both classical and modern (in the applied sense) operational methods for the analysis of lumped, linear, dynamical systems as exemplified by electric circuits.

In the first chapter the author establishes the circuit concept in relation to Maxwell's electromagnetic field equations leading then to the formulation of the mesh and nodal methods of network analysis. The second chapter is concerned with the classical solution of the network integro-differential equations with particular attention to initial conditions and discontinuous excitation functions. This chapter also treats the Superposition Principle leading naturally to the concept of indicial admittance and Carson's infinite integral theorem. Chapter Three is a discussion of the circuit analogues of mechanical and thermal systems, which is indeed a valuable addition. Chapter Four returns again to the solution of the circuit problem, this time utilizing the Heaviside-Jeffreys operational calculus. Recent trends in engineering education have tended to ignore Heaviside's operator in favor of the Laplace transform. This is not necessarily regrettable, although it does appear to remove the student one step further from physical reality. At least one cogent argument in favor of including the operational calculus is the fact that it appears throughout much important engineering literature. The Laplace transform method, which to many readers will be the "main plot" of this volume, is very clearly expounded in Chapter Five. In the final chapter the author discusses the concept of the frequency spectrum, starting with the Fourier series and showing the transition to the Fourier integral. The ideas expressed in this chapter are almost indispensable for a clear understanding of the operations involved when transform methods are employed, and might well have come earlier in the book.

There is no doubt that this volume will be favorably received by applied mathematicians, physicists, and engineers. The style is concise but lucid, and with many clear diagrams makes for pleasurable reading. A very valuable feature is the copious references to the literature. This book is unusually well documented. A large number of problems are included at the end of each chapter, some of which will challenge the better students. *R. Kahal.*

Soldi, Mario. Soluzione di problemi relativi a reti lineari col metodo delle equazioni ai nodi. Alta Frequenza **18**, 213–231 (1949).

A brief and precise statement of the nodal method for lumped-constant parameter networks is followed by several generalizations. In place of the usual equivalent circuits for triode and pentode electron tubes, it is suggested that unilateral admittances be used. An alternative treatment of transformer equivalent circuits based on the introduction of an additional node is also presented. A number of examples are given in detail and applications to quadripoles and electronic circuits are presented. The methods are based upon a direct interpretation of the nodal equations in circuit terms. *C. Saltzer* (Cleveland, Ohio).

Yablonskiĭ, S. V. Realization of a linear function in a class of Π-schemes. Doklady Akad. Nauk SSSR (N.S.) **94**, 805–806 (1954). (Russian)

A bound due to J. Riordan [See C. E. Shannon, Bell System Tech. J. **28**, 59–98 (1949); these Rev. **10**, 671] for the number of contacts required for the series-parallel realization of the linear Boolean functions $c_0+c_1x_1+\cdots+c_nx_n$ (mod 2) is established. *C. Saltzer* (Cleveland, Ohio).

Tsang, N. F. On electrical network determinants. J. Math. Physics **33**, 185–193 (1954).

A theorem is given stating that under certain topological conditions the ratio of the determinant of the mesh equations to the determinant of the nodal equations is the product of the branch admittances. This theorem is proved for passive networks. For a general class of networks (non-degenerate) it is shown that the zeros of the mesh and nodal determinants coincide. For networks with no coupled branches it is shown that the sum of the ratios of the branch impedance to the two-terminal impedance "looking in" is equal to the number of meshes, while the corresponding sum for the admittances is shown to be the number of nodes minus one. *C. Saltzer* (Cleveland, Ohio).

Quantum Mechanics

***Iwanenko, D., und Sokolow, A. Klassische Feldtheorie.** Akademie-Verlag, Berlin, 1953. xii+347 pp. DM 19.00.

Translation of the authors' "Klassičeskaya teoriya polya" [Gostehizdat, Moscow-Leningrad, 1949; these Rev. **13**, 95].

Costa de Beauregard, Ol. La fin du conflit de la relativité et des quanta. Rev. Questions Sci. (5) **15**, 317–335, 481–503 (1954).

Bopp, Fritz. Korpuskularstatistische Begründung der Quantenmechanik. Z. Naturforschung **9a**, 579–600 (1954).

An entirely new foundation of quantum mechanics, indpendent of the concepts and equations of classical mechanics, is developed on the basis of purely statistical considerations and without occurrence of the conventional wave-particle complementarity. Particles are considered as objects perfectly localized in ordinary three-dimensional space at all times, but identical particles under identical experimental conditions, initially localized at the same point, will usually follow different orbits. Exact reproducibility of time dependence is only achieved for statistical assemblies of particles: the set W of statistical distribution functions defining the state of the assembly at a given instant t uniquely determines the corresponding set W' defining the state of the assembly at any other time t': $W'=L(W)$. The transformation L is shown to be linear by considering the reunion of two statistical assemblies as one new such assembly. The properties of L are discussed with particular

reference to reversibility of the motion. It is established that W can always be represented mathematically by an hermitian matrix P of trace 1, and that one should expect the transformation $W'=L(W)$ to take in this representation the form $P'=U^{+}PU$, with U a unitary matrix. The connection with the usual quantum-mechanical formalism is thus obtained, P playing the role of the density matrix. The Schrödinger equation is finally derived on the basis of invariance considerations. *L. Van Hove* (Utrecht).

Iwata, Giiti. A formulation of field theory in Hilbert space. Progress Theoret. Physics 11, 537–556 (1954).

On the basis of general considerations presented in earlier papers [same journal 6, 684–690 (1951); 7, 39–44 (1952); these Rev. 13, 806, 1010] the author formulates field theory by considering the field variables as vectors or tensors in a Hilbert space, and by submitting them to a hyperquantization invariant for unitary transformations in this Hilbert space. The method is applied to quantum electrodynamics. The formulae obtained for pair creation, pair annihilation and Compton scattering coincide with the usual ones in lowest order. There is no vacuum polarization. The electron self-energy diverges and cannot be removed by the usual renormalization method. *L. Van Hove* (Utrecht).

Thermodynamics, Statistical Mechanics

Ichimura, Hiroshi. A method in quantum statistical mechanics. I. General theory. Progress Theoret. Physics 11, 374–384 (1954).

The quantum-mechanical grand partition function of a system of interacting particles is expanded in powers of the interaction energy by means of the second quantization formalism. Part of the series obtained is summed in exponential form. No study, however, is made of the importance of the terms not included in this summation.
L. Van Hove (Utrecht).

Ichimura, Hiroshi. A method in quantum statistical mechanics. II. The degenerate Fermi-Dirac assembly. Progress Theoret. Physics 11, 385–391 (1954).

The formulae of the preceding paper are used for a fairly crude estimate of the effect of interactions between electrons on the thermodynamic properties of conduction electrons. The interaction is represented by a screened-off Coulomb (i.e. a Yukawa) potential. *L. Van Hove* (Utrecht).

Ichimura, Hiroshi. A method in quantum statistical mechanics. III. Virial expansion for the quantum gas. Progress Theoret. Physics 11, 519–527 (1954).

The expansion of the quantum-mechanical grand partition function of a system of particles in powers of the interaction potential, derived in the paper reviewed second above, is here used for the determination of the second virial coefficient of quantum gases. The result is applied to very low temperatures, although no justification is given for the clearly doubtful use of an expansion in powers of the interaction in the neighborhood of absolute zero.
L. Van Hove (Utrecht).

MacDonald, William M., III, and Richardson, John M. Approximate variational principle in quantum statistics. Physical Rev. (2) 96, 18–21 (1954).

The variational principle of quantum statistics is formulated in terms of the density matrix. The formalism of second quantization is applied for an arbitrary choice of the basic orthonormal set of single particle states, and the assumption is introduced that the density matrix is a product of factors each of which is a function of the occupation number of one single particle state. The variational principle is applied under this assumption, variations being performed for the single particle states as well as for the factors of the density matrix. Equations of the Hartree-Fock type are obtained for systems of interacting Fermi or Bose particles at arbitrary temperature. *L. Van Hove.*

Lindhard, J. On the properties of a gas of charged particles. Danske Vid. Selsk. Mat.-Fys. Medd. 28, no. 8, 57 pp. (1954).

The behaviour of a gas (possibly degenerate) of charged particles is discussed with attention concentrated on the equations for the electromagnetic field in the gas. First the electron gas is discussed semi-classically by a perturbation treatment of the Boltzmann equation, assuming the existence of a relaxation time. The microscopic electromagnetic field is treated classically and transverse and longitudinal dielectric constants are derived. Next the electron gas is described quantum-mechanically. Limiting values of the ratios of the lengths, viz., wavelength of electron, wavelength of field, distance travelled by electron in one period of the field, give simple interpretations. Applications are given to the slowing down of a charged particle in the gas, and to straggling and multiple small-angle scattering in energy loss. The relation to other discussions is indicated.
C. Strachan (Aberdeen).

Callen, Herbert B., Barasch, Murray L., and Jackson, Julius L. Statistical mechanics of irreversibility. Physical Rev. (2) 88, 1382–1386 (1952).

The authors extend their "fluctuation-dissipation" theorem to the case of several fluctuating variables. By a quantum-mechanical perturbation procedure, the response of a system to a small amplitude periodic disturbance is derived, yielding an admittance matrix, $g_{ij}(\omega)$. Then, computing the correlation function of two randomly fluctuating variables in equilibrium, it is found to be simply connected with the admittance matrix and energy density $E(\omega, T)$,

$$\langle Q_j Q_k \rangle = \frac{2}{\pi}\int_0^\infty E(\omega, T)[g_{jk}(\omega)+g_{kj}(\omega)]\frac{d\omega}{2\omega^2},$$

$$E(\omega, T)=\tfrac{1}{2}\hbar\omega+\hbar\omega[\exp(\hbar\omega/kT)-1]^{-1}.$$

Arguments concerned with time-reversal yield the symmetry relation $Y_{jk}(\omega, A)=Y_{kj}(\omega, -A)$ where $Y_{jk}=g_{jk}-ib_{jk}$ and A is the vector potential of a magnetic field, if present. This relation is more general than Onsager's in applying to each frequency ω separately, but is different in applying to periodic forcing terms rather than to a steady state. The extension of these results to the latter case is indicated.
H. Grad (New York, N. Y.).

Greene, Richard F., and Callen, Herbert B. On a theorem of irreversible thermodynamics. II. Physical Rev. (2) 88, 1387–1391 (1952).

[For part I see Physical Rev. (2) 86, 702–710 (1952); these Rev. 14, 230.] Results similar to those of the above reviewed paper are derived by a 'macroscopic', thermodynamic analysis. It is assumed that the system is undergoing a stationary stochastic process in certain macroscopic variables to which standard thermodynamic formalism (in particular fluctuation theory) can be applied; there exists a

linear relation between the Fourier transforms of a driving force and response; correlation functions are even in time (microscopic reversibility); finally, the expectation of ξ_j at time τ knowing $\xi_0' \cdots \xi_r'$ at time 0, $\langle \tau, \xi_0' \cdots \xi_r' | \xi_j \rangle$, is assumed to be identical with the macroscopically observed decay of the parameter ξ_j from the initial state $\xi_0' \cdots \xi_r'$. Since the admittance is directly related to this macroscopic decay, a relation between admittance and the autocorrelation function (i.e. fluctuations) follows immediately from

$$\langle \xi_i(t)\xi_j(t+\tau)\rangle = \int W_1(\xi_0' \cdots \xi_r')\langle \tau, \xi_0' \cdots \xi_r' | \xi_j \rangle;$$

W_1 is the equilibrium distribution of $\xi_0' \cdots \xi_r'$. Finally, the Wiener-Khinchin formula

$$G_{ij}(\omega) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int e^{-i\omega\tau}\langle \xi_i(t)\xi_j(t+\tau)\rangle d\tau$$

yields the spectral density. Explicit expressions are presented for various forms of constraint, microcanonical, canonical, and adiabatic. Only the symmetry of the real part of the admittance matrix is shown. *H. Grad.*

ter Haar, D., and Green, C. D. The statistical aspect of Boltzmann's H-theorem. Proc. Phys. Soc. Sect. A. **66**, 153–159 (1953).

The author considers certain special problems as illustrative of the mechanism of irreversibility and the approach to equilibrium. The first model is represented by N numbered balls distributed between two urns. At uniform intervals a ball is chosen at random and transferred to the other urn. The second model is similar but with a Bernouilli distribution for the number of balls transferred during each time interval. A measure of the departure from equilibrium (entropy) is introduced, and it is shown that for large N there is a strong tendency for approach to equilibrium. Recurrence times are also computed. *H. Grad.*

Wergeland, Harald. On the principle of least dissipation of energy. Norske Vid. Selsk. Forh., Trondheim **24** (1951), 110–113 (1952).

Consider the dissipation function

$$\Phi = \int \tfrac{1}{2} \sum X_j \cdot J_j \, dV,$$

where $J_i = \sum a_{ij} X_j$, (*) $X_j = F_j K - \text{grad}\,(\mu_j/T)$, J_1 = heat flow, $F_1 = 0$, $\mu_1 = 1$, as a functional of the potentials $\mu_j(x)$ and temperature $T(x)$, keeping $a_{ij}(x)$ (symmetric, positive-definite), $F_j(x)$ and the boundary values of μ_i and T fixed. Vanishing of the first variation of Φ implies

$$(**) \qquad \begin{aligned} &\text{div } J_i = 0, \quad i \neq 1, \\ &\text{div } J_1 - \sum F_j \cdot J_j = 0, \end{aligned}$$

which are the conditions for a steady state, i.e. no material or energy flow. Since Φ is positive definite, we conclude that a steady state with the given boundary values of μ_i and T minimizes Φ. This principle is compared with the variational principle of Onsager, in which the roles of the side condition (*) and Euler equation (**) are interchanged.

H. Grad (New York, N. Y.).

Morgenstern, Dietrich. General existence and uniqueness proof for spatially homogeneous solutions of the Maxwell-Boltzmann equation in the case of Maxwellian molecules. Proc. Nat. Acad. Sci. U. S. A. **40**, 719–721 (1954).

A simple proof is given for the existence of a unique solution of the Maxwell-Boltzmann equation under the conditions that the solution be spatially homogeneous for all times with given initial conditions and that molecular forces be of the Maxwell type. The proof is based upon Wild's iterative procedure [Proc. Cambridge Philos. Soc. **47**, 602–609 (1951); these Rev. **13**, 195] for obtaining a monotonic sequence of approximations. This sequence is shown to have bounded integrals and so converge for all t to an integrable limit function satisfying the Boltzmann equation. Wild had previously proved convergence only for a finite time interval. It is also shown that the proof can be extended to a somewhat larger class of molecular interactions including the Maxwell molecules. This class is defined by a functional equation, the physical interpretation of which is not too clear. This, along with Carleman's proof for the existence of solutions of another special type, are the only such existence theorems that have been proved for this equation. *G. Newell* (Providence, R. I.).

Widom, B. The virial series of the ideal Bose-Einstein gas. Physical Rev. (2) **96**, 16–17 (1954).

Explicit expressions are derived for the virial coefficients of the ideal Bose-Einstein gas but the asymptotic values could not be evaluated. Several lower bounds for the radius of convergence of the virial expansion are found and the author conjectures that the radius of convergence is infinite even though the above lower bounds are all well below the known condensation density. *G. Newell.*

Falkenhagen, H., und Kelbg, G. Klassische Statistik unter Berücksichtigung des Raumbedarfs der Teilchen. II. Ann. Physik (6) **14**, 391–396 (1954).

Glauberman, A. E., and Yuhnovskiĭ, I. R. On the "superposition" approximation in the theory of systems of interacting particles. Doklady Akad. Nauk SSSR (N.S.) **93**, 999–1002 (1953). (Russian)

Starting from a set of equations for the coordinate distribution functions of a system of interacting particles in statistical equilibrium proposed by N. N. Bogolyubov [Problems of dynamical theory in statistical physics, Gostehizdat, Moscow-Leningrad, 1946; these Rev. **13**, 196], the authors discuss an approximate method of solving these equations. This involves an expansion in powers of a parameter and also makes use of the "superposition" approximation, in which, for example, the distribution function for three particles is written as the product of the distribution functions for the three possible pairs of particles.

N. Rosen (Haifa).

Shimose, Ikuro. Statistical theory of phase transition. I. General theory for lattice gas. J. Phys. Soc. Japan **9**, 451–456 (1954).

The properties of a two-dimensional lattice gas are discussed on the basis of intuitive consideration of clusters of molecules and holes with various sizes and properties. The occurrence of a phase transition is described in terms of such clusters. *L. Van Hove* (Utrecht).

Shimose, Ikuro. Statistical theory of phase transition. II. Numerical calculation for lattice gas. J. Phys. Soc. Japan **9**, 457–459 (1954).

The considerations of the preceding paper are used for an approximate numerical determination of the equation of state of a two-dimensional lattice gas. The few results given are in excellent agreement with the values obtained from the exact treatment. *L. Van Hove* (Utrecht).

Ikenberry, Ernest. On the distribution function in phase space, with application to the statistical theory of liquids. J. Chem. Phys. 19, 467–470 (1951).

The author introduces into Liouville's equation an expansion in Hermite functions of the molecular velocity components and obtains an infinite sequence of partial differential equations for the Hermite coefficients as functions of position and time. Results are stated for the values of the coefficients of heat conductivity and viscosity of a liquid in terms of time constants for the decay of a local disturbance in temperature and velocity. These time constants are expressed in terms of the mean curvature of the mean intermolecular potential. *H. Grad* (New York, N. Y.).

Ikenberry, Ernest. On the distribution function in phase space, with application to the stationary state of an ideal gas. J. Chem. Phys. 20, 568–570 (1952).

The formalism described in the above reviewed paper is applied to a stationary state of an ideal gas (no intermolecular forces) in a gravitational field. This serves only to illustrate the method since Liouville's equation is solvable explicitly in this case (the molecules execute parabolic trajectories). *H. Grad* (New York, N. Y.).

Van Hove, Léon. Correlations in space and time and Born approximation scattering in systems of interacting particles. Physical Rev. (2) 95, 249–262 (1954).

Whereas Born approximation elastic scattering formulas can be expressed in terms of the density of particles in the scattering system and nearly elastic scattering in terms of the pair distribution function $g(\mathbf{r})$ of the scattering system, the formulas for strongly inelastic scattering are somewhat more complicated. This report shows, however, that even in this case the relevant features of the scattering can be described in terms of a function $G(\mathbf{r}, t)$ which is a generalization of the function $g(\mathbf{r})$.

In the absence of quantum-mechanical effects, $G(\mathbf{r}, t)$ corresponds to the average density distribution as seen at time t from a point at which a particle was present at time zero; thus, for $t=0$, $G(\mathbf{r}, 0)=g(\mathbf{r})+\delta(\mathbf{r})$. The complete definition and the general properties of $G(\mathbf{r}, t)$ are described and its significance discussed particularly in relation to scattering of slow neutrons. The qualitative behavior of $G(\mathbf{r}, t)$ is described for dense gases and liquids and explicit formulas are given for a vibrating crystal lattice and for an ideal quantum gas. The author expresses the hope that experimental neutron scattering data will in the near future be able to give significant information about $G(\mathbf{r}, t)$ for various physical systems which in turn would produce much desired data on statistical correlations in these systems.

G. Newell (Providence, R. I.).

Roy, S. K. On the evaluation of certain lattice series. Canadian J. Physics 32, 509–514 (1954).

This paper is concerned with the summation of certain lattice series in the form

$$\sum(-1)^{x+y+z}(x^m+y^m+z^m)/(x^2+y^2+z^2)^{m+l}$$

where m is an integer and the summation is taken over a set of triples of integers associated with some lattice configuration. These series are conditionally convergent and Evjen [Physical Rev. (2) 39, 675–687 (1932)] has proposed a method (in the case $m=0$) of summing by combining terms associated with lattice subcells which have as many positive as negative terms. The author applies this process in two cases in which a non-unique result is obtained and suggests the additional criterion that the cells be chosen so that any regrouping of these cells cannot lead to a separation of the positive and negative terms into layers of similar terms. He shows how this can be done for the case of a face-centered cubical lattice or body-centered cubical lattice. *F. J. Murray* (New York, N. Y.).

Montroll, Elliott W. Frequency spectrum of vibrations of a crystal lattice. Proceedings of the symposium on special topics in applied mathematics, Northwestern University, 1953. Amer. Math. Monthly 61, no. 7, part II, 46–75 (1954).

This is a review of some of the recent developments in the analysis of the vibration spectrum of crystal lattices with particular emphasis on the observation that the frequency distribution function is expected to have singularities. The paper by Van Hove [Physical Rev. (2) 89, 1189–1193 (1953); these Rev. 15, 88], which gives sound arguments for the existence of logarithmic singularities for almost all two-dimensional crystals and vertical tangent singularities for three-dimensional crystals, is given a less rigorous but simplified interpretation.

Numerous examples are used to illustrate the existence and number of singularities, particularly for anisotropic crystals. The illustrations, many of which are considered here for the first time, are carefully chosen for their unusual mathematical simplicity and do not have much direct physical application; in fact, a large part of them are two-dimensional examples. The singularities considered are due to saddle points in the surface describing the frequency as a function of the wave vector. The author fails, however, to point out that there are also singularities caused by local maximum and minimum as well as saddle points of this surface. Some of the graphs do not, for this reason, show all the singular points of the spectrum. *G. Newell.*

***Thompson, Sanford P. Statistical mechanics of complex vibrating systems.** Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 49–56. The American Society of Mechanical Engineers, New York, N. Y., 1952.

The author uses the methods of statistical mechanics to describe the vibrations of a dynamical system of many degrees of freedom. The Hamiltonian is taken quadratic in coordinates and in momenta, i.e. the normal modes are weakly or non-interacting. Two probability distributions are obtained, one canonical in the total energy of the system, the other a product of canonical distributions with different temperature for each mode. Some applications are mentioned. *H. Grad* (New York, N. Y.).